SOLUTION FOR NOVEMBER 2016

Prove that the polynomial $p(x) = x^3 - 12x^2 + ax - 64$ has all of its roots real and nonnegative for exactly one real number a. Determine a.

SOLUTION:

$$a = 48 \text{ and thus } p(x) = (x-4)^3.$$

Let $p(x) = x^3 - 12x^2 + ax - 64$ and let us assume that $p(x)$ has all of its roots real and nonnegative. So we can write:

$$p(x) = (x - r_1)(x - r_2)(x - r_3) \text{ with } 0 \leq r_1 \leq r_2 \leq r_3.$$

Multiplying this out and equating coefficients gives:

$$r_1 + r_2 + r_3 = 12,\ r_1r_2 + r_1r_3 + r_2r_3 = a,\ r_1r_2r_3 = 64.$$

From the equation $r_1r_2r_3 = 64$ we see that in fact that r_1, r_2, r_3 must be strictly positive and then using this in the equation $r_1r_2 + r_1r_3 + r_2r_3 = a$ implies that $a > 0$.

Next we see that:

$$p'(x) = 3x^2 - 24x + a.$$

If $p'(x) \geq 0$ for all x then p has only one real root and so it must be that $r_1 = r_2 = r_3$ and therefore $3r_1 = 12$ and so $r_1 = r_2 = r_3 = 4$ and thus $48 = 16 + 16 + 16 = a$ and so we are done in this case. So now suppose $p'(x)$ gets negative. Then since p' is a quadratic then $p'(x) = 0$ has two real solutions. From the quadratic formula we see that $p'(x) = 0$ at:

$$4 \pm \sqrt{\frac{48-a}{3}}.$$

Thus we must have $48 - a \geq 0$, i.e. $a \leq 48$. Thus we see $0 < a \leq 48$. Further $p(x)$ has a local maximum at $4 - \sqrt{\frac{48-a}{3}}$ and a local minimum at $4 + \sqrt{\frac{48-a}{3}}$. Next we rewrite $p(x)$ as:

$$p(x) = (x-4)^3 + (a-48)x.$$

Then we calculate and see that if $0 < a < 48$ then:

$$p\Big(4 - \sqrt{\frac{48-a}{3}}\Big) = \frac{2(48-a)}{3}\Big[\sqrt{\frac{48-a}{3}} - 6\Big] < 0 \text{ since } 0 < a < 48.$$

Then it follows that $p(x)$ has only one real root and as earlier this implies $r_1 = r_2 = r_3 = 4$ and thus $48 = 16 + 16 + 16 = a$. This contradicts that $0 < a < 48$ and therefore our assumption that $0 < a < 48$ must be false. From earlier we know $a > 0$ so it must be that $a \geq 48$. From earlier we also know $a \leq 48$ therefore it must be that $a = 48$. This completes the proof.