SOLUTION FOR JANUARY 2017 PROBLEM

Let $A = (0, 2)$ and $B = (3, 0)$. Find a point, C, on the circle of radius 1 centered at the origin that makes a triangle ABC of largest possible area.

Correct solutions turned in by William Liu and Xiangyu Kong.

SOLUTION:

$$C = (-\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}).$$

Let us denote (x_0, y_0) as the point C and note that $x_0^2 + y_0^2 = 1$. Also note that the distance from A to B is $\sqrt{13}$. So the area of the triangle is going to be: $\frac{1}{2}\sqrt{13}h$ where h is the height. Here h is the length of the line segment from (x_0, y_0) to the line through points A and B. The line through A and B is: $y = -\frac{2}{3}x + 2$. And the line through (x_0, y_0) that is perpendicular to y is given by $Y - y_0 = \frac{3}{2}(x - x_0)$ which is $Y = \frac{3}{2}x - \frac{3}{2}x_0 + y_0$. These line intersect when:

$$x = \frac{1}{13}(9x_0 - 6y_0 + 12)$$

and:

$$y = \frac{1}{13}(-6x_0 + 4y_0 + 18).$$

Note that:

$$x - x_0 = \frac{2}{13}(-2x_0 - 3y_0 + 6)$$

and:

$$y - y_0 = \frac{3}{13}(-2x_0 - 3y_0 + 6).$$

Thus the square of the height of the triangle we are interested in is:

$$h^2 = (x - x_0)^2 + (y - y_0)^2 = \frac{1}{13}(-2x_0 - 3y_0 + 6)^2.$$

Thus:

$$h = \frac{1}{\sqrt{13}}(-2x_0 - 3y_0 + 6).$$

And from earlier the area of the triangle is:

$$Area = \frac{1}{2}\sqrt{13}h = \frac{1}{2}(-2x_0 - 3y_0 + 6).$$

So we just need to find the maximum of this function. Denoting $x_0 = \cos(t)$ and $y_0 = \sin(t)$ then we need to maximize:

$$\frac{1}{2}(-2\cos(t) - 3\sin(t) + 6).$$

Differentiating and setting equal to zero gives:

$$2\sin(t) - 3\cos(t) = 0$$

so:

$$\frac{y_0}{x_0} = \tan(t) = \frac{2}{3}, \ i.e. \ y_0 = \frac{2}{3}x_0.$$

And since $x_0^2 + y_0^2 = 1$ we obtain $x_0^2 = \frac{9}{13}$ and $y_0^2 = \frac{4}{13}$. So either $x_0 = \frac{3}{\sqrt{13}}$, $y_0 = \frac{2}{\sqrt{13}}$ or $x_0 = -\frac{3}{\sqrt{13}}$, $y_0 = -\frac{2}{\sqrt{13}}$. It should now be clear from the geometry that the max occurs when:

$$x_0 = -\frac{3}{\sqrt{13}}, y_0 = -\frac{2}{\sqrt{13}}.$$