SOLUTION FOR FEBRUARY 2017 PROBLEM

Let $P = (-3/2, 9/4)$ and let $Q = (3, 9)$. Note that P and Q both lie on $y = x^2$. Let $y = f(x)$ be a continuous function on the interval $[-3/2, 3]$ and suppose the function $f(x)$ lies above the line PQ. Find a point R on the curve $y = x^2$ between P and Q so that the area bounded by $y = f(x)$ and the straight line segments PR and QR is as large as possible.

SOLUTION:

$$R = (3/4, 9/16).$$

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Denote $R = (x, x^2)$. Let y_1 be the line from PR i.e. let $y_1(t)$ be the line from $P = (-3/2, 9/4)$ to $R = (x, x^2)$. A short computation shows: $y_1(t) = 9/4 + (x - 3/2)(t + 3/2)$ where $-3/2 \le t \le x$. Simlarly, let $y_2(t)$ be the line from R to Q i.e. from (x, x^2) to $(3, 9)$. Then $y_2(t) = 9 + (x+3)(t-3)$ where $x \le t \le 3$. We want to find the maximum of the function:

$$A(x) = \int_{-3/2}^{x} [f(t) - y_1(t)]\, dt + \int_{x}^{3} [f(t) - y_2(t)]\, dt.$$

Notice we may rewrite this as:

$$A(x) = \int_{-3/2}^{3} f(t) - \left(\int_{-3/2}^{x} y_1(t)\, dt + \int_{x}^{3} y_2(t)\, dt \right).$$

Since the first integral is a constant and does not depend on x we see that the *maximum* of $A(x)$ occurs at the *minimum* of:

$$B(x) = \int_{-3/2}^{x} y_1(t)\, dt + \int_{x}^{3} y_2(t)\, dt.$$

It is now straightforward to show:

$$\int_{-3/2}^{x} y_1(t)\, dt = (1/2)x^3 + (3/4)x^2 + (9/8)x + (27/16)$$

and:

$$\int_{x}^{9} y_2(t)\, dt = -(1/2)x^3 + (3/2)x^2 - (9/2)x + (27/2).$$

Thus:

$$B(x) = (9/4)x^2 - (27/8)x + (243/16).$$

This is a parabola and the minimum occurs when $B'(x) = 0$ i.e. when $(9/2)x - (27/8) = 0$. Thus $x = 3/4$ and hence $x^2 = 9/16$.