SOLUTION FOR APRIL 2017 PROBLEM

Determine the area of the largest equilateral triangle that can be inscribed inside a square with side of length 1.

SOLUTION: The largest area is $2\sqrt{3} - 3$ and it is obtained when one of the vertices of the triangle is in a corner and the other two lie on adjacent edges.

First let us see that the largest the area can be occurs when the vertices are on the edge of the square. If none of the vertices lie on the edge then we can always scale the triangle slightly. That is, if s is the length of one of the sides of the triangle, then multiply the length of each side by $k > 1$. Since each vertex is originally interior to the square then if $k > 1$ but k is sufficiently close to 1 then the new triangle will have sides of length greater than s (and hence have larger area) and will still lie within the unit square. Thus for the equilateral triangle with maximum area at least one of the vertices has to lie on the edge of the square. See Figure 1.

So now suppose one vertex lies on the edge but the other two do not. Again we can scale the triangle slightly by multiplying each side by $k > 1$ and increase the area. Thus at least two of the vertices must lie on an edge. See Figure 2.

Now suppose that two vertices lie on the edge but the third does not. Then there are two possibilities - either the vertices on the edge lie on adjacent edges of the square or they lie on opposite sides of the square. In the first case we can translate the triangle so that all three vertices lie inside the square then as before we can scale the triangle slightly and the new triangle will have a larger area. This rules out this possibility. See Figure 3. Now suppose two vertices lie on opposite sides of the square and the third vertex does not lie on an edge. Then notice that it is always possible to rotate the triangle about one of the vertices on the edge so that the other vertex on the edge is now interior to the square. Then we have only one vertex on the edge and this case was ruled out already. See Figure 4.

Thus all three vertices must lie on the edge to obtain the largest area. Next we show that one of the vertices must be in a corner so suppose not. Then it is always possible to rotate the triangle slightly so that less than 3 vertices lie on an edge which we know from earlier is not possible. See Figure 5. Thus one vertex must lie in a corner.

Now let us label the square as $ABCD$ and suppose that one vertex is at A. Then if we construct an equilateral triangle with vertex at A and the other two vertices on the edge of the square then the other two vertices must be on BC and CD. Denote P as the vertex on BC and Q as the vertex on CD. Then we notice that triangle ABP and AQD are right triangles with angles 15^0 and 75^0. Since the length of AB is 1 it then follows that AP has length $\sec(15^0) = \sqrt{6} - \sqrt{2} = \sqrt{2}(\sqrt{3} - 1)$. Thus the area of the equilateral triangle is:

$$\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}2(4 - 2\sqrt{3}) = 2\sqrt{3} - 3.$$