SOLUTION FOR DECEMBER 2018

Solution:

$$\theta = \frac{3\pi}{4}.$$

Proof: Denote the angle between the sides of length 1 and 2 as θ and the angle between the sides of length 2 and 3 as ϕ. Then the remaining angle is $2\pi - \theta - \phi$.

Applying the law of cosines to the three triangles gives:

$$x^2 = 5 - 4\cos(\theta) \tag{1}$$

$$x^2 = 13 - 12\cos(\phi) \tag{2}$$

and since $\cos(x)$ is even and 2π periodic:

$$2x^2 = 10 - 6\cos(2\pi - \theta - \phi) = 10 - 6\cos(\theta + \phi). \tag{3}$$

Thus from (3) we get:

$$x^2 = 5 - 3\cos(\theta + \phi). \tag{4}$$

Then from (1) and (4) we see:

$$3\cos(\theta + \phi) = 4\cos(\theta). \tag{5}$$

Also equating (1) and (2) we get:

$$3\cos(\phi) = 2 + \cos(\theta). \tag{6}$$

Now using the addition formula for cosine in (5) and rewriting gives:

$$\cos(\theta)(3\cos(\phi) - 4) = 3\sin(\theta)\sin(\phi).$$

And using (6) in this gives:

$$\cos(\theta)(\cos(\theta) - 2) = 3\sin(\theta)\sin(\phi). \tag{7}$$

Now squaring both sides and using $\sin^2(x) + \cos^2(x) = 1$ gives:

$$\cos^4(\theta) - 4\cos^3(\theta) + 4\cos^2(\theta) = (1 - \cos^2(\theta))(9 - 9\cos^2(\phi)).$$

Using (6) on the right-hand side of this gives:

$$\cos^4(\theta) - 4\cos^3(\theta) + 4\cos^2(\theta) = 5 - 4\cos(\theta) - 6\cos^2(\theta) + 4\cos^3(\theta) + \cos^4(\theta).$$

Thus:

$$8\cos^3(\theta) - 10\cos^2(\theta) - 4\cos(\theta) + 5 = 0$$

so:

$$(2\cos^2(\theta) - 1)(4\cos(\theta) - 5) = 0.$$

Now $\cos(\theta) = 5/4 > 1$ so this equation has no solutions and so:

$$\cos(\theta) = \pm\frac{1}{\sqrt{2}}$$

therefore:

$$\theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}.$$

Now if $\theta = \frac{\pi}{4}$ then it follows from (6) that $\cos(\phi) > 0$ and so $0 < \phi < \frac{\pi}{2}$. Then $\sin(\phi) > 0$ so the left-hand side of (7) is negative but the right-hand side of (7) is positive which is impossible so $\theta \neq \frac{\pi}{4}$. Thus $\theta = \frac{3\pi}{4}$.