SOLUTION FOR JANUARY 2018

Let a, b, c be real numbers such that $ax^2 + 2bxy + cy^2 > 0$ for all $(x, y) \neq (0, 0)$.

Determine:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^{-(ax^2+2bxy+cy^2)} \right) dx\, dy.$$

SOLUTION:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^{-(ax^2+2bxy+cy^2)} \right) dx\, dy = \frac{\pi}{\sqrt{ac - b^2}}.$$

It follows from the fact that $ax^2 + 2bxy + cy^2 > 0$ for all $(x, y) \neq (0, 0)$ that $ac - b^2 > 0$. Then rewriting and completing the square we obtain:

$$ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{ac - b^2}{a}y^2.$$

Thus:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^{-(ax^2+2bxy+cy^2)} \right) dx\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^{-a(x+\frac{b}{a}y)^2 - \frac{(ac-b^2)}{a}y^2} \right) dx\, dy$$

$$= \int_{-\infty}^{\infty} e^{-\frac{(ac-b^2)}{a}y^2} \left(\int_{-\infty}^{\infty} e^{-a(x+\frac{b}{a}y)^2}\, dx \right) dy. \quad (1)$$

Changing variables in the innermost integral by letting $w = x + \frac{b}{a}y$ gives:

$$\int_{-\infty}^{\infty} e^{-a(x+\frac{b}{a}y)^2}\, dx = \int_{-\infty}^{\infty} e^{-aw^2}\, dw = \sqrt{\frac{\pi}{a}}. \quad (2)$$

Inserting (2) into (1) gives:

$$\sqrt{\frac{\pi}{a}} \int_{-\infty}^{\infty} e^{-\frac{(ac-b^2)}{a}y^2}\, dy = \sqrt{\frac{\pi}{a}} \sqrt{\frac{\pi}{\frac{ac-b^2}{a}}} = \frac{\pi}{\sqrt{ac - b^2}}.$$