SOLUTION FOR MAY 2018

Let $a > 0$ and $b > 0$. Determine:

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt.$$

SOLUTION:

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt = 2\pi.$$

Since the integrand is 2π periodic we can integrate over any interval of length 2π so:

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt.$$

Next since $\cos^2(t+\pi) = \cos^2(t)$ and $\sin^2(t+\pi) = \sin^2(t)$ it follows that:

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt.$$

Also since $\cos^2(t)$ and $\sin^2(t)$ are even functions it follows that:

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt = 4\int_0^{\frac{\pi}{2}} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt.$$

Next dividing top and bottom by $\cos^2(t)$ we obtain:

$$4\int_0^{\frac{\pi}{2}} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\, dt = 4\int_0^{\frac{\pi}{2}} \frac{ab\sec^2(t)}{a^2 + b^2\tan^2(t)}\, dt.$$

Now let $u = \tan(t)$ so $du = \sec^2(t)\, dt$ thus:

$$4\int_0^{\frac{\pi}{2}} \frac{ab\sec^2(t)}{a^2 + b^2\tan^2(t)}\, dt = 4\int_0^{\infty} \frac{ab}{a^2 + b^2u^2}\, du$$

$$= \lim_{u\to\infty} 4\tan^{-1}\left(\frac{b}{a}u\right) = 4\frac{\pi}{2} = 2\pi.$$