## SOLUTION FOR JANUARY 2019

Solution:

$$A = 1 + \frac{\pi}{3} - \sqrt{3}.$$

**Proof:** First we see that the area of the square is 1 so in terms of $A, B$ and $C$ we obtain:

$$A + 4B + 4C = 1. \tag{1}$$

Next the area of a quarter circle with radius 1 is $\frac{\pi}{4}$ and so in terms of $A, B$ and $C$ we obtain:

$$A + 3B + 2C = \frac{\pi}{4}. \tag{2}$$

Then using calculus we see that:

$$\frac{C}{2} = \text{ area under the circle } x^2 + (y-1)^2 = 1 \text{ on } [0, \frac{1}{2}].$$

Thus:

$$\frac{C}{2} = \int_0^{\frac{1}{2}} 1 - \sqrt{1 - x^2}\, dx. \tag{3}$$

From calculus:

$$\int_0^a \sqrt{1 - x^2}\, dx = \frac{1}{2}\left[\sin^{-1}(a) + a\sqrt{1 - a^2}\right].$$

Thus:

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx = \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{4}\right] = \frac{\pi}{12} + \frac{\sqrt{3}}{8}. \tag{4}$$

Therefore combining (3)-(4) we obtain:

$$C = 1 - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right). \tag{5}$$

Next multiplying (2) by 4, multiplying (1) by 3, and subtracting gives:

$$A = 4C + \pi - 3.$$

Finally using (5) we obtain:

$$A = 1 + \frac{\pi}{3} - \sqrt{3}.$$