SOLUTION FOR JANUARY 2020

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Find all functions which satisfy: $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$. (You may NOT assume f is continuous.)

Hint: Try to show f is nondecreasing.

SOLUTION:

$$f(x) = x \text{ or } f(x) = 0.$$

Proof: First it is not too hard to show just using $f(x + y) = f(x) + f(y)$ that $f(rx) = rf(x)$ for every rational number r.

First $f(0) = f(0 + 0) = f(0) + f(0)$ implies $f(0) = 0$.

Then $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$ and so $f(-x) = -f(x)$.

Next $f(2) = f(1+1) = f(1)+f(1) = 2f(1)$. And then by induction one can show $f(n) = nf(1)$ for every positive integer n. Then since $f(-n) = -f(n) = -f(1)n = f(1)(-n)$ we see $f(n) = nf(1)$ for all integers.

Next $f(1) = f(\frac{1}{2} + \frac{1}{2}) = f(\frac{1}{2}) + f(\frac{1}{2}) = 2f(\frac{1}{2})$ so $f(\frac{1}{2}) = \frac{1}{2}f(1)$. Similarly one can show $f(\frac{1}{3}) = \frac{1}{3}f(1)$ and more generally $f(r) = rf(1)$ for every rational r. And similarly one can show:

$$f(rx) = rf(x) \text{ for all rational } r.$$

For example, $f(2x) = f(x+x) = f(x)+f(x) = 2f(x)$ and $f(x) = f(\frac{1}{2}x)+f(\frac{1}{2}x) = 2f(\frac{1}{2}x)$. so $f(\frac{1}{2}x) = \frac{1}{2}f(x)$.

Next we use $f(xy) = f(x)f(y)$.
Let $x > y$ then $f(x) - f(y) = f(x - y) = f(\sqrt{x-y}\sqrt{x-y}) = \left[f(\sqrt{x-y})\right]^2 \geq 0$. Thus:

$$f(x) \geq f(y) \text{ if } x > y.$$

Now let x be an arbitrary real number and choose an increasing sequence of rationals p_n such that $\lim_{n\to\infty} p_n = x$. Then

$$f(x) \geq f(p_n) = p_n f(1) \to xf(1).$$

Thus we see

$$f(x) \geq xf(1).$$

Similarly choose a decreasing sequence of rationals q_n such that $\lim_{n\to\infty} q_n = x$.

$$f(x) \leq f(q_n) = q_n f(1) \to x f(1).$$

Thus

$$f(x) \leq x f(1)$$

and so we see $f(x) = f(1)x$. Finally, $[f(1)]^2 = f(1)f(1) = f(1 \cdot 1) = f(1)$ hence $f(1) = 0$ or $f(1) = 1$ and therefore either $f(x) = 0$ or $f(x) = x$. $\square$