SOLUTION FOR FEBRUARY 2020

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Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $f(0) = f(1)$. Show that there is an $0 < x_2 < 1$ such that $f(x_2 + \frac{1}{2}) = f(x_2)$. Similarly show there is an $0 < x_n < 1$ such that $f(x_n + \frac{1}{n}) = f(x_n)$.

Proof:

Let $g(x) = f(x + 1/n) - f(x)$ for $0 \le x \le (n-1)/n$. If $g(0) = 0$ then we are done. Otherwise suppose $g(0) > 0$. Then notice that:

$$g(0) + g(1/n) + g(2/n) + \cdots + g((n-1)/n) = f(1) - f(0) = 0.$$

Thus since $g(0) > 0$ then one of the other terms must be negative. Therefore for some smallest value of k with $1 \le k \le n-1$ we have $g(k/n) < 0$. Since f is continuous it follows that g is continuous and now we know $g(0) > 0$ and $g(k/n) < 0$. Therefore by the intermediate value theorem there is an x_0 such that $g(x_0) = 0$. Thus:

$$f(x_0 + 1/n) = f(x_0).$$

□