SOLUTION FOR APRIL 2020

Correct solutions were submitted by:

Show that for $0 \leq x \leq 1$:

$$(\sin^{-1}(x))^2 = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+2}}{(n+1)}. \tag{1}$$

Then use this to find:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}.$$

Hint: Show that $y = (\sin^{-1}(x))^2$ and the infinite sum satisfy the same linear second order differential equation and then solve it.

SOLUTION:

$$\frac{\pi^2}{18} = \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}.$$

Proof: Let:

$$w = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+2}}{(n+1)}.$$

First it is not hard to show using the ratio test that this series (and the series for w' and w'') converge for $|x| < 1$.

Notice that $w(0) = w'(0) = 0$. Then:

$$w' = \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+1)!} x^{2n+1},$$

$$w'' = \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n)!} x^{2n} = 2 + \sum_{n=0}^{\infty} \frac{2^{2n+3}((n+1)!)^2}{(2(n+1))!} x^{2n+2},$$

$$x^2 w'' = \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n)!} x^{2n+2},$$

and:

$$x w' = \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+1)!} x^{2n+2}.$$

Thus:

$$\begin{aligned}(1-x^2)w'' &= 2 + \sum_{n=0}^{\infty} \left(\frac{2^{2n+3}((n+1)!)^2}{(2(n+1))!} - \frac{2^{2n+1}(n!)^2}{(2n)!} \right) x^{2n+2} \\ &= 2 + \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+1)!} x^{2n+2} = 2 + x w'.\end{aligned}$$

Therefore:

$$(1-x^2)w'' - xw' = 2$$

which implies:

$$(\sqrt{1-x^2}\ w')' = \frac{2}{\sqrt{1-x^2}}.$$

Integrating gives:

$$\sqrt{1-x^2}\ w' = 2\sin^{-1}(x) + C.$$

Since $w'(0) = 0$ it follows that $C = 0$ and thus:

$$w' = \frac{2\sin^{-1}(x)}{\sqrt{1-x^2}}.$$

Integrating once more and using $w(0) = 0$ yields:

$$w = (\sin^{-1}(x))^2.$$

This establishes the first part of the problem.

Now let $x = 1/2$ in (1) and we obtain:

$$\frac{\pi^2}{36} = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!}\frac{1}{(n+1)2^{2n+2}} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}\frac{1}{(n+1)}$$

$$\frac{1}{4}\sum_{n=0}^{\infty}\frac{2((n+1)!)^2}{(2(n+1))!}\frac{1}{(n+1)^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{n^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}.$$

Thus:

$$\frac{\pi^2}{18} = \sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}.$$

□