SOLUTION FOR SEPTEMBER 2020

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Take a quarter circle with radius R in quadrant 1 with one side on the x-axis and one side on the y-axis. Take a semicircle of radius $0 < r < R$ and inscribe it in the larger circle so that the arc intersects and is tangent to the vertical and horizontal part of the quarter circle. Determine how r and R are related.

SOLUTION: with

$$r = \frac{R}{\sqrt{3}}.$$

Proof: Since the semicircle has radius r and the semicircular arc is tangent to the x-axis and y-axis it follows that if the semicircle is made into a circle then the center of the circle would be (r, r). Now the line through the origin and (r, r) bisects the semicircle and has slope 1. The segment that is the base of the semicircle is perpendicular to this line hence has slope -1 and goes through (r, r). Therefore it has equation $y = 2r - x$. Now we denote $P = (a, 2r - a)$ as the point which is at the end of the segment of the semicircle and for which $0 < a < r$. This point is r units from (r, r) and so from the distance formula we see that $\sqrt{(r-a)^2 + (r-a)^2} = r$ so $\sqrt{2}|r-a| = r$ and since $0 < a < r$ we see that $a = r(1 - \frac{1}{\sqrt{2}})$ and so we see $P = \left(r(1 - \frac{1}{\sqrt{2}}), r(1 + \frac{1}{\sqrt{2}})\right)$. Finally P is also on the quarter circle of radius R centered at the origin. Thus the distance from P to the origin is R hence $r^2(1 - \frac{1}{\sqrt{2}})^2 + r^2(1 + \frac{1}{\sqrt{2}})^2 = R^2$ so that $3r^2 = R^2$ and thus $r = \frac{R}{\sqrt{3}}$.