SOLUTION FOR JANUARY 2021

Correct solutions were submitted by:

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Find the equation of a circle $(x-h)^2+(y-k)^2=r^2$ that has exactly one rational point on it. Also find the equation of a circle that exactly two rational points on it.

SOLUTION: $(0,0)$ is the only rational point on $(x-\sqrt{2})^2+y^2=2$.

$(0,\pm 1)$ are the only rational points on $(x-\sqrt{2})^2+y^2=3$.

Proof: Suppose $(\frac{m}{n},\frac{p}{q})$ is a rational point on $(x-\sqrt{2})^2+y^2=2$ with m/n and p/q in lowest terms and $n\neq 0$ and $q\neq 0$. Suppose $m\neq 0$ and $p\neq 0$. Substituting in $(x-\sqrt{2})^2+y^2=2$ gives:

$$2\sqrt{2}q^2mn=q^2m^2+p^2n^2.$$

Notice the left-hand side is irrational since q,m,n are all nonzero and the right-hand side is rational thus we obtain a contradiction. Also subsitution yields that $(0,0)$ is on $(x-\sqrt{2})^2+y^2=2$. Thus the first statement has been proven.

Similarly, suppose $(\frac{m}{n},\frac{p}{q})$ is a rational point on $(x-\sqrt{2})^2+y^2=3$ with m/n and p/q in lowest terms and $n\neq 0$ and $q\neq 0$. Suppose $m\neq 0$ and $p\neq 0$. Substituting in $(x-\sqrt{2})^2+y^2=3$ gives:

$$2\sqrt{2}q^2mn=q^2m^2+(p^2-1)n^2.$$

Notice the left-hand side is irrational since q,m,n are all nonzero and the right-hand side is rational thus we obtain a contradiction. Also subsitution yields that $(0,\pm 1)$ are on $(x-\sqrt{2})^2+y^2=3$. Thus the second statement has been proven. $\square$