SOLUTION FOR FEBRUARY 2021

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Show that in any Pythagorean triangle one of the sides must be divisible by 3, one side must be divisible by 4, and one side must be divisible by 5.

SOLUTION: We first assume that (a, b, c) is a primitive Pythagorean triple and so it is known that there exist positive integers m, n with $m > n$ such that $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$.

Since (a, b, c) is primitive it follows that a and c are odd. In addition one of m, n is even for if m, n are odd then a, b, and c are all even contradicting that a and c are odd. Thus one of m, n is even and therefore since $b = 2mn$ then we see that b is a multiple of 4.

Next we show one is divisible by 3. Notice that we want to show that one of the following is divisible by 3: $m, n, m - n, m + n, m^2 + n^2$. So let us assume m and n are not divisible by 3. Then $m = 3k + m_1$ and $n = 3p + n_1$ where m_1 and n_1 are either 1 or 2. If $m_1 = n_1$ we see $m - n$ (hence a) is divisible by 3. And if $m_1 = 1$, $n_1 = 2$ or vice versa then we see $m_1 + n_1$ (hence a) is divisible by 3.

Next we show one is divisible by 5. Notice that we want to show that one of the following is divisible by 5: $m, n, m - n, m + n, m^2 + n^2$. So let us assume m and n are not divisible by 5. Then $m = 5k + m_1$ and $n = 5p + n_1$ where m_1 and n_1 are either $1, 2, 3$, or 4. If $m_1 = n_1$ then we $m - n$ (hence a) is divisible by 5 and if $m_1 + n_1 = 5$ then $m + n$ (hence a) is divisible by 5. So the only remaining cases are: (1) $m_1 = 1$ and $n_1 = 2$ or 3, (2) $m_1 = 4$ and $n_1 = 2$ or 3, (3) $m_1 = 2$ and $n_1 = 1$ or 4, (4) $m_1 = 3$ and $n_1 = 1$ or 4. Notice then in all of these cases that $m_1^2 + n_1^2$ (and hence c) is divisible by 5.

Finally an arbitrary Pythagorean triple is a multiple of a primitive Pythagorean triple and since the result holds for primitive ones then it holds for arbitrary ones.

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