SOLUTION FOR MARCH 2021

A correct solution was submitted by:

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Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes on every value exactly 3 times.

Note: It is known that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes on every value exactly 2 times.

SOLUTION: We define $f(x) = -x - 2$ on $[-2, -1)$, $f(x) = x$ on $[-1, 1]$. Let $f(x) = 2 - x$ on $(1, 2]$ and let $f(x) = x - 2$ on $(2, 4]$. We then extend f to the rest of the real line by defining $f(x + 6) = f(x) + 2$.

We now show that for each $y \in [0, 1]$ there are exactly three values $x_1 < x_2 < x_3$ such that $f(x_i) = y$. First suppose $y = 0$ and notice that $f(-2) = f(0) = f(2) = 0$. Next suppose $y = 1$ then notice that $f(1) = f(3) = f(5) = 1$. Now suppose $0 < y_0 < 1$ then observe that $f(y_0) = f(2 - y_0) = f(2 + y_0) = y_0$.

A similar argument works for $y \in [1, 2]$, for $y \in [2, 3]$, etc. In addition, f is odd so the argument also shows that the result holds for $y \leq 0$.