SOLUTION FOR OCTOBER 2022

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Determine all continuous functions which satisfy:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \text{ for all real } x, y.$$

SOLUTION: $f(x) = ax + b$ for some constants a, b.

Proof: First let $h(x) = f(x) - f(0)$. Then note that h is continuous and $h(0) = 0$. Also we see:

$$h\left(\frac{x+y}{2}\right) = f\left(\frac{x+y}{2}\right) - f(0) = \frac{f(x) - f(0) + f(y) - f(0)}{2} = \frac{h(x)+h(y)}{2}. \tag{1}$$

Now substituting $y = 0$ gives:

$$h\left(\frac{x}{2}\right) = \frac{h(x)}{2}.$$

It then follows from this that:

$$h\left(\frac{x+y}{2}\right) = \frac{h(x+y)}{2}.$$

Now using (1) we see that:

$$h(x+y) = h(x) + h(y).$$

It follows by induction that $h(nx) = nh(x)$ for all integers n and replacing x with $\frac{x}{n}$ gives $h(x/n) = (1/n)h(x)$. Now replacing x with mx where m is an integer and using the previous two equations gives $h(\frac{m}{n}x) = \frac{m}{n}h(x)$. Thus:

$$h(rx) = rh(x) \text{ for all rational } r. \tag{2}$$

Finally for a general real number r we can find a sequence of rationals r_n such that $\lim_{n\to\infty} r_n = r$. Now using (2) and the fact that h is continuous then it follows:

$$h(rx) = \lim_{n\to\infty} h(r_n x) = \lim_{n\to\infty} r_n h(x) = rh(x).$$

Recalling that $h(r) = f(r) - f(0)$, letting $x = 1$ and r an arbitrary real number gives $h(r) = rh(1) = [f(1) - f(0)]r$ and so we finally see:

$$f(x) = f(0) + [f(1) - f(0)]x.$$

□