SOLUTION FOR NOVEMBER 2022

Correct solutions were submitted by:

Ria Garg
Saisneha Ghatti
Matthew Li
Victor Lin (Runner-Up)
Eric Peng (Winner)

Of all triangles with base length b_0 and perimeter P_0 prove that the one that has maximum area must be isosceles.

Solution: Consider a triangle with sides of length b_0, a, c. Heron's formula for the area of a triangle is:

$$A = \sqrt{s_0(s_0 - a)(s_0 - b_0)(s_0 - c)}$$

where $s_0 = \frac{P_0}{2}$. We also know that $2s_0 = P_0 = b_0 + a + c$ and therefore $c = P_0 - b_0 - a$. Rewriting then gives:

$$A^2 = s_0(s_0 - b_0)(s_0 - a)(s_0 - (P_0 - b_0 - a)) = s_0(s_0 - b_0)(s_0 - a)(b_0 - s_0 + a).$$

The only variable here is a and so A^2 is a parabola which opens down and thus A^2 and hence A has an absolute maximum when the derivative of the right-hand side is 0. That is when $s_0 - b_0 + s_0 - 2a = 0$. Thus $2a = 2s_0 - b_0 = a + c$ and rewriting this then gives $c = a$. Thus the triangle must be isosceles. □