SOLUTION FOR DECEMBER 2022

Correct solutions were submitted by:

Saisneha Ghatti
Zachary Li - Runner-Up
Victor Lin - Winner
Eric Peng

Problem: Determine which is larger: $\sin(\cos(x))$ or $\cos(\sin(x))$.

Solution:

$$\cos(\sin(x)) > \sin(\cos(x)).$$

Proof: This solution is essentially the proof submitted by Zachary Li.

We first determine if $\cos(\sin(x))$ and $\sin(\cos(x))$ are ever equal. Notice since both of these functions is 2π periodic and even then it suffices to restrict to $x \in [0, \pi]$. So suppose:

$$\cos(\sin(x)) = \sin(\cos(x)) \text{ for } 0 \leq x \leq \pi. \tag{1}$$

Then using the trig identity $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$ on the left-hand side of (1) gives:

$$\sin\left(\sin(x) + \frac{\pi}{2}\right) = \sin(\cos(x)) \quad \text{i.e.} \quad \sin\left(\sin(x) + \frac{\pi}{2}\right) - \sin(\cos(x)) = 0. \tag{2}$$

Now recall the following trig identity:

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

It follows from this then that $\sin(x) = \sin(y)$ if and only if $x - y = 2k\pi$ or $x + y = (2k+1)\pi$ for some integer k. Thus if equation (2) holds then we must have: $\sin(x) + \frac{\pi}{2} - \cos(x) = 2k\pi$ or $\sin(x) + \frac{\pi}{2} + \cos(x) = (2k+1)\pi$. Now since $x \in [0, \pi]$ then we must have either:

$$\cos(x) - \sin(x) = \frac{\pi}{2}$$

or:

$$\cos(x) + \sin(x) = \frac{\pi}{2}.$$

Next one can check that:

$$-\sqrt{2} \leq \cos(x) - \sin(x) \leq \sqrt{2}$$

and:

$$-\sqrt{2} \leq \cos(x) + \sin(x) \leq \sqrt{2}$$

and since $\frac{\pi}{2} > \sqrt{2}$ then we see there are no solutions of $\cos(\sin(x)) = \sin(\cos(x))$.

Thus since both functions are continuous then we must have either $\cos(\sin(x)) > \sin(\cos(x))$ or $\cos(\sin(x)) < \sin(\cos(x))$. Finally since $\cos(\sin(0)) = 1$ and $\sin(\cos(0)) = \sin(1) < 1$ we see then that we must have:

$$\sin(\cos(x)) < \cos(\sin(x)).$$

□