SOLUTION FOR JANUARY 2022

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Find the isosceles triangle of smallest area that circumscribes a circle of radius $r > 0$.

HINT: It is known that if A is the area of the triangle and P the perimeter then $A = \frac{1}{2}rP$.

SOLUTION: The minimum area occurs when the triangle is equilateral and the minimum area is $A = 3\sqrt{3}r^2$.

PROOF: Denote the sides of the triangle as a, a, c and the height as h. Then by the Pythagorean theorem:

$$a^2 = h^2 + \frac{c^2}{4}. \tag{1}$$

Also from the hint we know $A = \frac{1}{2}rP$. Writing this out gives:

$$\frac{1}{2}ch = \frac{1}{2}r(2a + c).$$

So $ch = 2ra + rc$ and therefore $a = \frac{c(h-r)}{2r}$. Substituting this into (1) gives: $h^2 + \frac{c^2}{4} = \frac{c^2(h-r)^2}{4r^2}$ thus:

$$h^2 = \frac{c^2}{4}\frac{(h^2 - 2rh)}{r^2}$$

and so:

$$h = \frac{c^2}{4}\frac{(h - 2r)}{r^2}$$

hence:

$$h(c^2 - 4r^2) = 2rc^2. \tag{2}$$

It follows from this that $c > 2r$. Solving for h and substituting into the formula for the area gives:

$$A = \frac{1}{2}ch = \frac{rc^3}{c^2 - 4r^2}.$$

We now want to find the value of $c > 2r$ for which this function has a minimum. Notice that $\lim_{r \to 2c^+} A = \infty$ and $\lim_{c \to \infty} A = \infty$. Therefore A has a minimum on $(2r, \infty)$ and so differentiating with respect to c gives:

$$A' = \frac{rc^2(c^2 - 12r^2)}{(c^2 - 4r^2)^2}.$$

So the minimum occurs when $c^2 = 12r^2$ and thus $c = 2\sqrt{3}r$. Substituting into (2) gives $h = 3r$ and then using (1) we see $a = 2\sqrt{3}r = c$. Thus the minimum for A occurs when the triangle is equilateral and the minimum area is $A = \frac{1}{2}ch = \frac{1}{2}2\sqrt{3}r \cdot 3r = 3\sqrt{3}r^2$. $\square$