SOLUTION FOR FEBRUARY 2022

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Determine:

$$\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right)\frac{1}{n}.$$

(You may assume this series converges).

Hint: First show:

$$\frac{1}{\sqrt{1-4x}} = 1+\sum_{n=1}^{\infty}\binom{2n}{n}x^n \text{ for } |x|<\frac{1}{4}$$

and then:

$$2\ln\left(\frac{1-\sqrt{1-4x}}{2x}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n}{n}x^n \text{ for } |x|<\frac{1}{4}.$$

SOLUTION:

$$\ln(4) = \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdot\cdots(2n-1)}{2\cdot 4\cdot 6\cdot\cdots(2n)}\right)\frac{1}{n}.$$

Proof: The binomial series is just the Maclaurin series for $f(x) = (1+x)^\alpha$ where α is a real number. Writing this out gives:

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots.$$

It is straightforward to show using the ratio test that this series converges for $|x|<1$. Replacing x with $-4x$ and setting $\alpha=-\frac{1}{2}$ gives for $|x|<\frac{1}{4}$:

$$\frac{1}{\sqrt{1-4x}} = 1+2x+\frac{(1\cdot 3)2^2}{2!}x^2+\frac{(1\cdot 3\cdot 5)2^3}{3!}x^3+\cdots.$$

Next we observe the following:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{(1\cdot 3\cdot 5\cdot\cdots(2n-1))\,(2\cdot 4\cdot 6\cdots(2n))}{(n!)^2}$$

$$= \frac{(1\cdot 3\cdot 5\cdot\cdots(2n-1))\,2^n n!}{(n!)^2} = \frac{(1\cdot 3\cdot 5\cdot\cdots(2n-1))\,2^n}{n!} \tag{1}$$

Thus we see that:

$$\frac{1}{\sqrt{1-4x}} = 1+\sum_{n=1}^{\infty}\binom{2n}{n}x^n \text{ for } |x|<\frac{1}{4} \tag{2}$$

Next a straightforward but tedious exercise shows that:

$$\left(2\ln\left(\frac{1-\sqrt{1-4x}}{2x}\right)\right)' = \frac{1}{x}\left(\frac{1}{\sqrt{1-4x}}-1\right).$$

Thus it follows from (2) that:

$$\left(2\ln\left(\frac{1-\sqrt{1-4x}}{2x}\right)\right)' = \sum_{n=1}^{\infty}\binom{2n}{n}x^{n-1}$$

and therefore after integrating we see for some constant C:

$$2\ln\left(\frac{1-\sqrt{1-4x}}{2x}\right) = \sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{n} + C \text{ for } |x| < \frac{1}{4}.$$

Taking limits as $x \to 0^+$ shows that $C = 0$ and therefore:

$$2\ln\left(\frac{1-\sqrt{1-4x}}{2x}\right) = \sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{n} \text{ for } |x| < \frac{1}{4}.$$

Finally we take the limit as $x \to \frac{1}{4}^-$ and use (1) to obtain:

$$\ln(4) = 2\ln(2) = \sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{n4^n} = \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)\cdot 2^n}{n!}\frac{1}{n4^n}$$

$$= \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{n2^n n!} = \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\dots\cdot(2n)}\right)\frac{1}{n}.$$

One final note - showing that:

$$\lim_{x\to\frac{1}{4}^-}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{n} = \sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{n4^n} \text{ which is equivalent to: } \lim_{x\to 1^-}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{4^n n} = \sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{n4^n}$$

is not immediately obvious but it does follow from Abel's limit theorem which states that if $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ and:

$$\lim_{x\to 1^-}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

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