SOLUTION FOR MARCH 2022

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Consider a trapezoid with parallel sides, A, B of length a and b. Draw the diagonals and label their intersection as I. Now draw the line that is parallel to A and B that goes through I. Determine the length of this segment, L, in terms of a and b.

SOLUTION:

$$L = \frac{2ab}{a+b}.$$

Proof: We place side A on the x-axis with one end at $(0,0)$ and the other at $(a,0)$. The parallel side B we place at height $y = d$ and label its endpoints (c,d) and $(b+c,d)$ (since side B has length b). The diagonal through $(0,0)$ and $(b+c,d)$ is given by $y = \frac{dx}{b+c}$ and the diagonal through (c,d) and $(a,0)$ is given by $y = \frac{d(a-x)}{a-c}$. We see then that these two lines intersect at $(\frac{a(b+c)}{a+b}, \frac{ad}{a+b})$. Then we see that the points on the segment which is parallel to A and B and that lie on the trapezoid are $(\frac{ac}{a+b}, \frac{ad}{a+b})$ and $(\frac{a(2b+c)}{a+b}, \frac{ad}{a+b})$. Finally we see the length of this segment is:

$$L = \frac{a(2b+c)}{a+b} - \frac{ac}{a+b} = \frac{2ab}{a+b}.$$

Note: Given $a > 0$ and $b > 0$, their *harmonic mean* is defined to be $H(a,b) = \frac{2ab}{a+b} = \frac{1}{\frac{1}{2}(\frac{1}{a}+\frac{1}{b})}$. Notice that $H(a,b)$ is the reciprocal of the arithmetic mean of $1/a$ and $1/b$.