SOLUTION FOR SEPTEMBER 2022

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Let x_i be real numbers with $x_1 < x_2 < \cdots < x_{n-1} < x_n$. Determine the value of x for which the following function has a minimum:

$$|x - x_1| + |x - x_2| + \cdots + |x - x_{n-1}| + |x - x_n|.$$

SOLUTION: If n is odd then $n = 2k - 1$ and the minimum occurs at $x = x_k$. If n is even then $n = 2k$ and the minimum occurs at all x with $x_k \le x \le x_{k+1}$.

Proof: First note that $f(x) = |x - x_1| + |x - x_2| + \cdots + |x - x_{n-1}| + |x - x_n|$ is continuous for all x and differentiable at all x except $x = x_1, x = x_2, \cdots, x = x_n$.

Also observe that f is a piecewise linear function. That is, the graph of f is just made up of a set of line segments and the slope of the line segments increases as x increases.

Next notice that for $x \ge x_n$ then $f(x) = (x - x_1) + (x - x_2) + \cdots + (x - x_{n-1}) + (x - x_n) = nx - (x_1 + x_2 + \cdots x_n)$ and therefore f is just a line segment with slope n.

Next notice that for $x_{n-1} \le x \le x_n$ then $f(x) = (x - x_1) + (x - x_2) + \cdots + (x - x_{n-1}) - (x - x_n) = (n-2)x - (x_1 + x_2 + \cdots x_{n-1} - x_n)$ and therefore f is just a line segment with slope $n - 2$.

Similarly for $x_{n-2} \le x \le x_{n-1}$ then $f(x)$ is another line segment with slope $n - 4$, etc.

Therefore if n is even with $n = 2k$ then $f(x)$ will be a horizontal line segment (and hence slope 0) for $x_k \le x \le x_{k+1}$ and so the minimum of f occurs at all x with $x_k \le x \le x_{k+1}$.

Now if n is odd with $n = 2k - 1$ then on the segment (x_k, x_{k+1}) f will have slope 1 and on the segment $(x_{k-1}, x_k))$ f will have slope -1 and therefore f will have a unique minimum at $x = x_k$. $\square$