SOLUTION FOR NOVEMBER 2023

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Prove for any positive integer N that:

$$\sum_{k=1}^{n} \frac{k(k+1)(k+2)\cdots(k+N-1)}{N!} = \frac{n(n+1)(n+2)\cdots(n+N)}{(N+1)!}$$

Solution: We prove this by induction on n. When $n = 1$ there is only one term in the sum on the left-hand side and we see this is:

$$\frac{1 \cdot 2 \cdot \cdots \cdot N}{N!} = \frac{N!}{N!} = 1$$

and the right-hand side is:

$$\frac{1 \cdot 2 \cdots (N+1)}{(N+1)!} = \frac{(N+1)!}{(N+1)!} = 1.$$

Thus we see the equation holds for $n = 1$.
We now assume:

$$\sum_{k=1}^{n} \frac{k(k+1)(k+2)\cdots(k+N-1)}{N!} = \frac{n(n+1)(n+2)\cdots(n+N)}{(N+1)!}$$

and try to prove:

$$\sum_{k=1}^{n+1} \frac{k(k+1)(k+2)\cdots(k+N-1)}{N!} = \frac{(n+1)(n+2)\cdots(n+N)(n+1+N)}{(N+1)!}.$$

So we begin with the left-hand side and divide it into two terms and use our inductive hypothesis to get:

$$\sum_{k=1}^{n+1} \frac{k(k+1)(k+2)\cdots(k+N-1)}{N!} = \sum_{k=1}^{n} \frac{k(k+1)(k+2)\cdots(k+N-1)}{N!} + \frac{(n+1)(n+2)(n+3)\cdots(n+N)}{N!}$$

$$= \frac{n(n+1)(n+2)\cdots(n+N)}{(N+1)!} + \frac{(n+1)(n+2)(n+3)\cdots(n+N)}{N!}$$

$$= \frac{n(n+1)(n+2)\cdots(n+N)}{(N+1)!} + \frac{(n+1)(n+2)(n+3)\cdots(n+N)(N+1)}{(N+1)!}$$

$$= \frac{(n+1)(n+2)\cdots(n+N)(n+N+1)}{(N+1)!}.$$

□