PROBLEM FOR DECEMBER 2023

Let a, b, c be positive with $a + b + c = 1$. Determine the smallest value of:

$$f(a,b,c) = \left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right).$$

Solution: The minimum occurs when $a = b = c = \frac{1}{3}$ and thus:

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 2 \cdot 2 \cdot 2 = 8.$$

Proof: One way to do this is to solve for $c = 1 - a - b$, substitute into the $f(a,b,c)$, and find the minimum over all positive a, b with $0 < a + b < 1$.

However, another way to do this is to use Lagrange multipliers. So we define $g(a,b,c) = a + b + c - 1$ and we want to minimize $f(a,b,c)$ subject to the condition $g(a,b,c) = 0$. By the theory of Lagrange multipliers this occurs at a point where:

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c) \text{ for some constant } \lambda$$

that is when:

$$f_a = \lambda g_a, f_b = \lambda g_b, f_c = \lambda g_c. \tag{1}$$

Calculating these gives:

$$f_a = -\frac{1}{a^2}, f_b = -\frac{1}{b^2}, f_c = -\frac{1}{c^2}, g_a = g_b = g_c = 1$$

and so the extrema occur when:

$$-\frac{1}{a^2} = \lambda, -\frac{1}{b^2} = \lambda, \text{ and } -\frac{1}{c^2} = \lambda.$$

This implies $a^2 = b^2 = c^2$ and since a, b, c are assumed to be positive and $a + b + c = 1$ we see then that $a = b = c = \frac{1}{3}$ and $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 8$. Finally we notice that $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not a local maximum because for example we see that $f\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right) = \frac{25}{2} > 8$ and if some other point was a minimum then we would have been able to find another solution of the equations (1).