SOLUTION FOR JANUARY 2023

Correct solutions were submitted by:

Ria Garg
Saisneha Ghatti - Winner
Helen Li
Zachary Li - Runner-Up
Victor Lin

Determine the triangle of maximum area that can be inscribed in a circle of radius 1.

Solution: The triangle of maximum area that can be inscribed in a circle of radius 1 must be equilateral.

Proof: We will first show that the triangle must be isosceles and then show that it must be equilateral.

We denote the unit circle as the set of points in $\mathbb{R}^2$ satisfying: $x^2 + y^2 = 1$. We then inscribe a triangle in the circle.

Next we rotate the circle so that the base of the triangle is a horizontal segment. Let us denote the two points where this intersects the circle as (x_0, y_0) and $(-x_0, y_0)$ where $x_0 > 0$ and $x_0^2 + y_0^2 = 1$. We denote the third vertex of the triangle (x, y) where $x^2 + y^2 = 1$.

Now the area of this triangle is:

$$x_0(y - y_0).$$

Notice that x_0 and y_0 are not changing and we want to make $x_0(y - y_0)$ as large as possible where y is the only quantity that is varying (and also $x^2 + y^2 = 1$). So to make this as large as possible we must have $y = 1$ and then since $x^2 + y^2 = 1$ it follows that $x = 0$. Thus we see the triangle of maximum area has vertices $(x_0, y_0), (-x_0, y_0)$, and $(0, 1)$. Notice that the distance from $(0, 1)$ to (x_0, y_0) is the same as the distance from $(0, 1)$ to $(-x_0, y_0)$ and so the triangle of maximum area must be isosceles.

Next we show that the isosceles triangle of maximum area must be equilateral.

First we see that the area of the triangle is $A = x_0(1 - y_0) = x_0(1 \pm \sqrt{1 - x_0^2})$. Notice this is larger if the $'+'$ is chosen and so we now find the maximum of:

$$A = x(1 + \sqrt{1 - x^2}) = x + x\sqrt{1 - x^2}.$$

Then:

$$A' = 1 + \sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}}.$$

Thus setting $A' = 0$ and multiplying by $\sqrt{1 - x^2}$ gives:

$$\sqrt{1 - x^2} = 2x^2 - 1.$$

Squaring both sides and simplifying we obtain:

$$x^2(4x^2 - 3) = 0$$

so $x = 0, \frac{\sqrt{3}}{2}$. Note that $x = 1$ is also a critical point. Evaluating A at these three points gives: $A(0) = 0$, $A(1) = 1$, $A(\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{4}$ so the maximum is $\frac{3\sqrt{3}}{4}$ and the maximum occurs when $x = \frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$. It follows from this that the distance between any two vertices is $\sqrt{3}$ and hence we see that the maximum occurs when the triangle is equilateral. $\square$