SOLUTION FOR FEBRUARY 2023

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Prove that if all the coefficients in $ax^2 + bx + c = 0$ are odd integers then the roots of the equation cannot be rational.

Proof: Suppose $a = 2n + 1$, $b = 2m + 1$, and $c = 2p + 1$. Now the solutions are not rational if and only if $b^2 - 4ac$ is not a perfect square. So we assume by way of contradiction $b^2 - 4ac$ is a perfect square so that:

$$b^2 - 4ac = (2m + 1)^2 - 4(2n + 1)(2p + 1) = q^2. \tag{1}$$

Since b is odd then b^2 is odd and $4ac$ is even so it follows that q must be odd, say $q = 2r + 1$. Thus we see by rewriting (1) that:

$$4m^2 + 4m - 4(2n + 1)(2p + 1) = q^2 - 1 = 4r^2 + 4r.$$

Dividing by 4 gives:

$$m^2 + m - (2n + 1)(2p + 1) = r^2 + r = r(r + 1).$$

Thus:

$$m(m + 1) - r(r + 1) = (2n + 1)(2p + 1). \tag{2}$$

Now note that $m(m + 1)$ and $r(r + 1)$ are even and so it follows that the left hand side of (2) is even but the right hand side is odd and so we obtain a contradiction. Thus we see that $b^2 - 4ac$ cannot be a perfect square and therefore the solutions of $ax^2 + bx + c = 0$ must be irrational. $\square$