SOLUTION FOR OCTOBER 2024

A correct solution was submitted by:

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Find all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) + f(x)f(y) \tag{1}$$

SOLUTION The solutions are $f(x) = -1$ or $f(x) = -1 + e^{cx}$ for some some real number c.

Proof First we let $g(x) = 1 + f(x)$. Then it follows from (1) that

$$g(x+y) = 1 + f(x+y) = 1 + f(x) + f(y) + f(x)f(y) = (1 + f(x))(1 + f(y)) = g(x)g(y).$$

Thus

$$g(x+y) = g(x)g(y). \tag{2}$$

Next note that $g(x) = g(\frac{x}{2} + \frac{x}{2}) = g(\frac{x}{2})g(\frac{x}{2}) = g^2(\frac{x}{2}) \geq 0$ and so $g(x) \geq 0$. Next if $g(x_0) = 0$ for some x_0 then $g(x) = g(x_0)g(x - x_0) = 0$ for all x and therefore $f(x) = -1$ for all x. Now we suppose $g(x) > 0$ for all x. Then let $h(x) = \ln(g(x))$ and we see from (2) that

$$h(x+y) = \ln(g(x+y)) = \ln(g(x)g(y)) = \ln(g(x)) + \ln(g(y)) = h(x) + h(y).$$

Thus

$$h(x+y) = h(x) + h(y). \tag{3}$$

and we also know h is continuous.

It is then well-known that $h(x) = h(1)x$. (First note that $h(0) = h(0+0) = h(0) + h(0)$ and so $h(0) = 0$. Next note that $h(2) = h(1) + h(1) = 2h(1)$. With a bit more work one can show that $h(r) = h(1)r$ for all rational r and then since h is assumed to be continuous then for general x write $x = \lim_{n\to\infty} r_n$ where the r_n are rational and so by continuity of h we have $h(x) = \lim_{n\to\infty} h(r_n) = \lim_{n\to\infty} h(1)r_n = h(1)x$).

So we now have

$$h(1)x = h(x) = \ln(g(x)) \text{ and so } g(x) = e^{h(1)x} \text{ thus } f(x) = -1 + e^{h(1)x}.$$

Letting $c = h(1)$ we see then that

$$f(x) = -1 + e^{cx}.$$

Thue we see that the solutions are

$$f(x) = -1 \text{ or } f(x) = -1 + e^{cx} \text{ for some real } c.$$