SOLUTION FOR NOVEMBER 2024

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Consider the square centered at the origin $[-1,1]\times[-1,1]$. Let S be the set of all (x,y) such that (x,y) is closer to the origin than (x,y) is from the boundary of the square. Determine the area of region S.

The area of S is $\frac{4}{3}(4\sqrt{2}-5)$.

Proof We examine the region in the first quadrant and then multiply the answer by 4 to get the area of S. Consider now (x,y) in the first quadrant with $y \geq x$ and (x,y) on the boundary of S.
Then we see that

$$\sqrt{x^2+y^2} = (1-y)$$

so squaring and rewriting we get:

$$y = \frac{1-x^2}{2}.$$

Now if (x,y) is on the diagonal, i.e. $y = x$, then the formula reads $x = \frac{1-x^2}{2}$ and thus $x^2+2x-1 = 0$ so since we are assuming $x > 0$ then we have $x = -1+\sqrt{2}$. So the region S in the first quadrant with $y \geq x$ is the region satisfying $0 \leq y \leq \frac{1-x^2}{2}$ where $0 \leq x \leq -1+\sqrt{2}$.
Next we examine the boundary of the region in the first quadrant with $y \leq x$. Then we have

$$\sqrt{x^2+y^2} = (1-x)$$

and thus $y^2 = 1-2x$ and $-1+\sqrt{2} \leq x \leq \frac{1}{2}$. So $y = \sqrt{1-2x}$ and $-1+\sqrt{2} \leq x \leq \frac{1}{2}$ thus for $-1+\sqrt{2} \leq x \leq \frac{1}{2}$ the region S in the first quadrant satisfies: $0 \leq y \leq \sqrt{1-2x}$. Finally then the area of S is:

$$4\left(\int_0^{-1+\sqrt{2}} \frac{1-x^2}{2}\,dx + \int_{-1+\sqrt{2}}^{\frac{1}{2}} \sqrt{1-2x}\,dx\right).$$

Let us temporarily denote $b = -1+\sqrt{2}$ and recall that $b^2 = 1-2b$.
Now we compute (and use that $b^2 = 1-2b$) to obtain

$$4\left(\int_0^{b} \frac{1-x^2}{2}\,dx + \int_{b}^{\frac{1}{2}} \sqrt{1-2x}\,dx\right) = 4\left(\frac{b}{2} - \frac{b^3}{6} + \frac{1}{3}(1-2b)^{\frac{3}{2}}\right)$$

$$= 4\left(\frac{b}{2} - \frac{b^3}{6} + \frac{b^3}{3}\right) = 4\left(\frac{b}{2} + \frac{b^3}{6}\right) = 2b\left(1+\frac{b^2}{3}\right) = \frac{4}{3}(4\sqrt{2}-5).$$