SOLUTION FOR SEPTEMBER 2024

The only correct solution was submitted by

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Let a, b be positive. Prove that:

$$G(a,b) = G\left(\frac{a+b}{2}, \sqrt{ab}\right)$$

where:

$$G(a,b) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{(x^2+a^2)(x^2+b^2)}}\, dx.$$

Proof: We change variables and set:

$$x = \frac{1}{2}\left(u - \frac{ab}{u}\right) \text{ where } 0 < u < \infty.$$

Then $-\infty < x < \infty$ and:

$$dx = \frac{1}{2}\left(1 + \frac{ab}{u^2}\right) du = \frac{u^2+ab}{2u^2}\, du.$$

Also:

$$\sqrt{x^2+ab} = \frac{1}{2}\left(u + \frac{ab}{u}\right) = \frac{u^2+ab}{2u}$$

and:

$$\sqrt{x^2 + \left(\frac{a+b}{2}\right)^2} = \frac{\sqrt{u^2+a^2}\sqrt{u^2+b^2}}{2u}.$$

Then we see:

$$G\left(\frac{a+b}{2}, \sqrt{ab}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{(x^2+(\frac{a+b}{2})^2)(x^2+ab)}}\, dx$$

$$= 2\int_0^{\infty} \frac{1}{\sqrt{(u^2+a^2)(u^2+b^2)}}\, du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{(u^2+a^2)(u^2+b^2)}}\, du = G(a,b).$$

□

Note: Given $a_0 > b_0 > 0$ we define:

$$a_{n+1} = \frac{a_n + b_n}{2}; b_{n+1} = \sqrt{a_n b_n}.$$

This is called the *arithmetic-geometric mean sequence* of a_0 and b_0. It turns out it can be shown that a_n is decreasing and b_n is increasing for $n \geq 1$ and in fact there is a $c > 0$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$!!! Thus from the problem just solved we see that:

$$G(a_{n+1}, b_{n+1}) = G(\frac{a_n+b_n}{2}, \sqrt{a_n b_n}) = G(a_n, b_n)$$

and therefore:

$$G(a_n, b_n) = G(a_0, b_0) \text{ for all } n \geq 1.$$

Further since $G(a, b)$ is continuous in both a and b then taking the limit as $n \to \infty$ we obtain:

$$G(c, c) = G(a_0, b_0)$$

but $G(c, c)$ is easy to compute and it is just $G(c, c) = \frac{\pi}{c}$. Thus we see that:

$$c = \frac{\pi}{G(a_0, b_0)}.$$

The number c is often denoted $c = AGM(a_0, b_0)$.