SOLUTION FOR MARCH 2025

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Solution

$$\int_0^\infty e^{-(x^2+\frac{1}{x^2})}\,dx = \frac{\sqrt{\pi}}{2e^2}.$$

(You may assume $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.)

Proof

Denote

$$I = \int_0^\infty e^{-(x^2+\frac{1}{x^2})}\,dx. \tag{1}$$

First making the change of variable $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$ in (1) gives:

$$I = -\int_\infty^0 e^{-(u^2+\frac{1}{u^2})}\frac{1}{u^2}\,du = \int_0^\infty \frac{1}{x^2}e^{-(x^2+\frac{1}{x^2})}\,dx. \tag{2}$$

Adding (1)-(2) gives:

$$2I = \int_0^\infty \left(1+\frac{1}{x^2}\right)e^{-(x^2+\frac{1}{x^2})}\,dx.$$

Now let $u = x - \frac{1}{x}$, $du = (1+\frac{1}{x^2})\,dx$, $u^2+2 = x^2+\frac{1}{x^2}$ and we obtain

$$2I = \int_{-\infty}^\infty e^{-u^2-2}\,du = e^{-2}\int_{-\infty}^\infty e^{-u^2} = e^{-2}\sqrt{\pi}.$$

Therefore we see

$$I = \int_0^\infty e^{-(x^2+\frac{1}{x^2})}\,dx = \frac{\sqrt{\pi}}{2e^2}.$$