SOLUTION FOR MARCH 2026

The following students turned in correct solutions:

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Problem:

Prove

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = \frac{\pi^2}{6}.$$

It will be helpful to know this famous formula discovered by the great mathematician Euler

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots . \tag{1}$$

Proof: First we note that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test. It follows from this that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges absolutely and uniformly for $|x| \le 1$.

Next recall if $|x| < 1$ then $\frac{1}{1-x} = 1 + x + x^2 + \cdots$. Replacing x with xy we obtain that if $|xy| < 1$ then

$$\frac{1}{1-xy} = 1 + xy + x^2y^2 + \cdots .$$

Now we integrate in y on $[0, b]$ where $0 < b < 1$ and we obtain

$$\int_0^b \frac{1}{1-xy}\, dy = \left(y + \frac{xy^2}{2} + \frac{x^2y^3}{3} + \cdots \right)\Big|_0^b = b + \frac{xb^2}{2} + \frac{x^2b^3}{3} + \cdots . \tag{2}$$

Now integrate this with respect to x on $[0, c]$ where $0 < c < 1$ and use (2) to obtain

$$\int_0^c \int_0^b \frac{1}{1-xy}\, dx\, dy = \left(xb + \frac{x^2b^2}{2^2} + \frac{x^3c^3}{3^2} + \cdots \right)\Big|_0^c$$

$$= bc + \frac{(bc)^2}{2^2} + \frac{(bc)^3}{3^2} + \cdots \le 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \tag{3}$$

Now taking limits as $b \to 1^-$ and $c \to 1^-$ we see that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy \le \frac{\pi^2}{6}.$$

Next we know from earlier that $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2}$ converges absolutely and uniformly for $|x| \leq 1$ and therefore from (3) as $b \to 1^-$ and $c \to 1^-$ we have

$$\frac{\pi^2}{6} \geq \int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy \geq \int_0^c \int_0^b \frac{1}{1-xy}\, dx\, dy = bc + \frac{(bc)^2}{2^2} + \frac{(bc)^3}{3^2} + \cdots \to 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Thus

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

$\square$