SOLUTION FOR APRIL 2026

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Problem:

The *Catalan numbers* are defined by $C_0 = 1$ and

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0. \tag{1}$$

Let

$$F(x) = C_0x + C_1x^2 + C_2x^3 + \cdots = \sum_{n=0}^{\infty} C_n x^{n+1} \tag{2}$$

and assume this series converges for small values of x. Prove that

$$F(x) = \frac{1 - \sqrt{1-4x}}{2}.$$

Hint: Show that $F(x)$ satisfies a quadratic equation.

Solution: Since we are assuming that the series for $F(x)$ converges for small values of x and since this is a power series it follows that the series converges absolutely for small value of x. Then by a theorem from analysis, if $G(x) = \sum_{n=0}^{\infty} g_n x^n$ and $H(x) = \sum_{n=0}^{\infty} h_n x^n$ converge absolutely for $|x| < R$ then $G(x)H(x) = \sum_{n=0}^{\infty} j_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} g_k h_{n-k} \right) x^n$ converges absolutely for $|x| < R$. Next taking $G(x) = H(x) = F(x)$ and using (1)-(2) we obtain

$$\begin{aligned}
F^2(x) &= (C_0x + C_1x^2 + C_2x^3 + \cdots)(C_0x + C_1x^2 + C_2x^3 + \cdots) \\
&= C_0^2x^2 + (C_0C_1 + C_1C_0)x^3 + (C_0C_2 + C_1C_1 + C_2C_0)x^4 + \cdots \\
&= C_1x^2 + C_2x^3 + C_3x^4 + \cdots \\
&= F(x) - C_0x
\end{aligned}$$

and since $C_0 = 1$ we obtain

$$F^2(x) = F(x) - x.$$

Thus

$$F^2(x) - F(x) + x = 0.$$

By the quadratic formula we see

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2} \text{ and since } F(0) = 0 \text{ we must choose the } - \text{ sign so } F(x) = \frac{1-\sqrt{1-4x}}{2}.$$

Next, using the binomial series it is tedious but straightforward to show

$$\frac{1}{\sqrt{1-4x}} = 1 + \sum_{n=1}^{\infty} \binom{2n}{n} x^n.$$

Integrating both sides of this on $(0, x)$ gives

$$-\frac{1}{2}\sqrt{1-4x} = -\frac{1}{2} + x + \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1}$$

and thus

$$F(x) = \frac{1-\sqrt{1-4x}}{2} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1}.$$

Finally comparing with (2) we see $C_n = \frac{1}{n+1}\binom{2n}{n}$.