SOLUTION FOR OCTOBER 2012

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Problem Determine

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{j=1}^{2n} (n^2 + j^2)^{\frac{1}{n}}.$$

SOLUTION:

$$\lim_{n\to\infty} \frac{1}{n^4} \left(\prod_{j=1}^{2n} (n^2 + j^2)^{\frac{1}{n}} \right) = e^{\frac{1}{2}\int_0^4 \ln(1+\frac{x^2}{4})\,dx} = \frac{25}{e^4} e^{2\tan^{-1}(2)}$$

Proof First we rewrite this as follows

$$\frac{1}{n^4} \prod_{j=1}^{2n} \left[(n^2 + j^2)^{\frac{1}{n}} \right] = \frac{1}{n^4} \prod_{j=1}^{2n} \left[n^{\frac{2}{n}} \left(1 + \frac{j^2}{n^2} \right)^{\frac{1}{n}} \right] = \prod_{j=1}^{2n} \left(1 + \frac{j^2}{n^2} \right)^{\frac{1}{n}}. \quad (1)$$

Now taking logs of both sides gives

$$\ln \left(\frac{1}{n^4} \prod_{j=1}^{2n} \left[(n^2 + j^2)^{\frac{1}{n}} \right] \right) = \left(\frac{1}{n} \sum_{j=1}^{2n} \ln(1 + \frac{j^2}{n^2}) \right). \quad (2)$$

This resembles a Riemann sum so let us recall if $f(x)$ is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^{n} f\Big(a + \frac{j(b-a)}{n}\Big).$$

Taking $a = 0$, $b = 4$ gives

$$\int_0^4 f(x)\,dx = \lim_{n\to\infty} \frac{4}{n} \sum_{j=1}^{n} f\Big(\frac{4j}{n}\Big).$$

Let $f(x) = \ln(1 + \frac{x^2}{4})$ and we obtain

$$\int_0^4 \ln(1 + \frac{x^2}{4})\,dx = \lim_{n\to\infty} \frac{4}{n} \sum_{j=1}^{n} \ln(1 + \frac{4j^2}{n^2}).$$

Now replacing n with $2n$ and dividing by 2 gives

$$\frac{1}{2} \int_0^4 \ln(1 + \frac{x^2}{4})\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{2n} \ln(1 + \frac{j^2}{n^2}).$$

Thus returning to (2) and taking limits we see that

$$\lim_{n\to\infty} \ln \left(\frac{1}{n^4} \prod_{j=1}^{2n} \left[(n^2 + j^2)^{\frac{1}{n}} \right] \right) = \frac{1}{2} \int_0^4 \ln(1 + \frac{x^2}{4})\,dx.$$

Thus

$$\lim_{n\to\infty}\left(\frac{1}{n^4}\prod_{j=1}^{2n}\left[(n^2+j^2)^{\frac{1}{n}}\right]\right)=e^{\frac{1}{2}\int_0^4\ln(1+\frac{x^2}{4})\,dx}=\frac{25}{e^4}e^{2\tan^{-1}(2)}.$$

Note:

$$\frac{1}{2}\int_0^4\ln(1+\frac{x^2}{4})\,dx=\int_0^2\ln(1+x^2)\,dx=x\ln(1+x^2)-2x+2\tan^{-1}(x)|_0^2$$

$$=2\ln(5)-4+2\tan^{-1}(2)=\ln(\frac{25}{e^4})+2\tan^{-1}(2).$$