SOLUTION FOR NOVEMBER 2012

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = -\frac{\pi \ln 2}{2}.$$

SOLUTION: First make the substitution $x = 2y$, $dx = 2\, dy$ into the left-hand side of the above and then use the double angle formula for $\sin(x)$ to obtain

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = 2\int_0^{\frac{\pi}{4}} \ln(\sin 2y)\, dy = 2\int_0^{\frac{\pi}{4}} \ln(2\sin y \cos y))\, dy. \qquad (1)$$

Now using rules of logs we see that

$$\ln(2\sin y \cos y)) = \ln(2) + \ln(\sin y) + \ln(\cos y).$$

Substituting this into (1) gives

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = \frac{\pi \ln 2}{2} + 2\int_0^{\frac{\pi}{4}} \ln(\sin y)\, dy + 2\int_0^{\frac{\pi}{4}} \ln(\cos y)\, dy. \qquad (2)$$

Now use the trig identity $\cos y = \sin(\frac{\pi}{2} - y)$ in the second integral in (2) and then let $x = \frac{\pi}{2} - y$ to obtain

$$2\int_0^{\frac{\pi}{4}} \ln(\cos y)\, dy = 2\int_0^{\frac{\pi}{4}} \ln\sin(\frac{\pi}{2} - y)\, dy = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin x)\, dx. \qquad (3)$$

Substituting (3) into (2) gives

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = \frac{\pi \ln 2}{2} + 2\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$$

and so finally we see

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = -\frac{\pi \ln 2}{2}.$$