SOLUTION FOR SEPTEMBER 2012

Problem Determine those positive integers, n, for which $1+n+n^2+n^3+n^4$ is a perfect square.

Solution: The only integer for which this is true is when $n=3$ in which case $1+3+3^2+3^3+3^4=121=11^2$.

Proof: The strategy of the proof is to strictly sandwich the number $1+n+n^2+n^3+n^4$ between two consecutive squares and thus show the impossibility of $1+n+n^2+n^3+n^4$ being a square.

Case 1 n is even

We first look at the case when n is even and we notice (since $\frac{n}{2}$ is an integer) that

$$(n^2+\frac{n}{2})^2 = n^4+n^3+\frac{n^2}{4} < n^4+n^3+n^2+n+1.$$

Next we look at

$$(n^2+\frac{n}{2}+1)^2 = n^4+n^3+\frac{n^2}{4}+2(n^2+\frac{n}{2})+1$$

$$= n^4+n^3+n^2+n+1+\frac{7}{4}n^2 > n^4+n^3+n^2+n+1.$$

Thus we see that

$$(n^2+\frac{n}{2})^2 < n^4+n^3+n^2+n+1 < (n^2+\frac{n}{2}+1)^2.$$

Thus we see since $\frac{n}{2}$ is an integer, that $n^4+n^3+n^2+n+1$ is larger than $(n^2+\frac{n}{2})^2$ and it is less than the square of the very next integer $n^2+\frac{n}{2}+1$. Thus $n^4+n^3+n^2+n+1$ cannot be the square of an integer if n is even.

Case 2 n is odd

We first notice that if $n=1$ then $n^4+n^3+n^2+n+1=5$ which is not a perfect square so now we assume n is odd and $n \geq 3$.
Next since n is odd then $\frac{n-1}{2}$ and $\frac{n+1}{2}$ are even and so we notice the following

$$(n^2+\frac{n-1}{2})^2 = (n^4+n^3+n^2+n+1) - \frac{5}{4}n^2 - \frac{3}{2}n - \frac{3}{4} < n^4+n^3+n^2+n+1.$$

Also,

$$(n^2+\frac{n+1}{2})^2 = (n^4+n^3+n^2+n+1) + \frac{(n-3)(n+1)}{4} \geq (n^4+n^3+n^2+n+1).$$

Notice we get equality when $n=3$ but a strict inequality when $n>3$ and so when n is odd and $n>3$ then we see that $(n^4+n^3+n^2+n+1)$ is not a perfect square.