OCTOBER 2013 SOLUTION

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Problem Let

$$a_{n+1} = \frac{a_n^2}{a_n^2 - 3a_n + 3}$$

and $a_1 = 4$. Find the integer part of the 2013-th term of the sequence, that is the greatest integer smaller than or equal to a_{2013}.

SOLUTION The greatest integer smaller than or equal to a_{2013} is 3.

Proof Let

$$f(x) = \frac{x^2}{x^2 - 3x + 3}.$$

Then we may rewrite our sequence as $a_{n+1} = f(a_n)$.
Notice that $f(0) = 0, f(1) = 1, f(3) = 3, f(2) = 4, f(4) = \frac{16}{7} > 2$. Also note that $x^2 - 3x + 3$ is an irreducible quadratic and so $x^2 - 3x + 3 > 0$ for all x.
Also note that

$$f'(x) = \frac{-3x(x-2)}{(x^2 - 3x + 3)^2}.$$

Thus f has critical points at $x = 0$ and $x = 2$. In addition, $f'(x) < 0$ for $x > 2$ so f is decreasing for $x > 2$. Also notice that

$$f : [2, 4] \to [2, 4].$$

Also the first several terms of the sequence are

$$a_1 = 4, a_2 = 16/7 \approx 2.28, a_3 = 256/67 \approx 3.82, a_4 = 65536/27547 \approx 2.37.$$

Notice then that

$$a_2 < a_4 < a_3 < a_1.$$

We claim now that a_{2n} is increasing and a_{2n+1} is decreasing. To prove this we let $g(x) = f(f(x))$. Note that

$$a_{n+2} = g(a_n).$$

Also note that $g'(x) = f'(f(x))f'(x)$. And we know that $f'(x) \leq 0$ on $[2, 4]$. Also since $f : [2, 4] \to [2, 4]$ it follows that $2 \leq f(x) \leq 4$ and so $f'(f(x)) \leq 0$ on $[2, 4]$. Thus $g'(x) \geq 0$ on $[2, 4]$. Thus $a_3 < a_1$ implies $a_5 = g(a_3) \leq g(a_1) = a_3$ and similarly $\cdots \leq a_7 \leq a_5 \leq a_3 \leq a_1$. Thus the sequence a_{2n+1} is decreasing and bounded below by 2. Similarly $a_2 < a_4$ implies $a_4 = g(a_2) \leq g(a_4) = a_2$ and thus a_{2n} is increasing and bounded above by 4. Thus the sequence a_{2n} converges to some $L \leq 4$ and a_{2n+1} converges to some $L_1 \geq 2$. Since $a_{2n+1} = f(a_{2n})$ and f is continuous it follows that $L_1 = f(L)$ and since $a_{2n+2} = f(a_{2n+1})$ we also have $L = f(L_1)$. Solving these two equations gives $(L_1 - L)(3 + L + L_1 - LL_1) = 0$. One possibility is that $L_1 = L$. If $L_1 \neq L$ then $3 + L + L_1 - LL_1 = 0$ which implies $(L_1 - 1)(L - 1) = 4$ so $L_1 - 1 = \frac{4}{L-1}$. Going back to $L_1 = \frac{L^2}{L^2-3L+3}$ we $L_1 - 1 = \frac{3(L-1)}{L^2-3L+3}$. Thus $\frac{4}{L-1} = \frac{3(L-1)}{L^2-3L+3}$ so $4L_2 - 12L + 12 = 3(L^2 - 2L + 1)$ and so $L^2 - 6L + 9 = 0$ so $L = 3$ and then $L_1 = 3$. So we see that $L_1 = L$. So $L = \frac{L^2}{L^2-3L+3}$ and since we know $0 < 2 \leq L \leq 4$ this then gives $L^2 - 3L + 3 = L$ so $(L-1)(L-3) = L^2 - 4L + 3 = 0$. Since $2 \leq L \leq 4$ it follows that $L = 3$. Thus the a_{2n} increase to 3 and the a_{2n+1} decrease to 3. So we have

$$2 < 16/7 = a_2 < a_4 \leq \cdots a_{2n} \leq 3 \leq a_{2n+1} \leq \cdots a_3 < a_1 = 4.$$

Thus $3 < a_{2013} < 4$ and so the greatest integer smaller than or equal to a_{2013} is 3.