NOVEMBER 2013 SOLUTION

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Problem Suppose x is a real number and that $x^3 + x$ and $x^5 + x$ are rational. Is it possible for x to be irrational?

SOLUTION No, it is not possible for x to be irrational, and thus x must be rational.

Proof Without loss of generality we assume $x > 0$. Suppose x is irrational and

$$x(x^2 + 1) = x^3 + x = \frac{a}{b}$$

and

$$x(x^4 + 1) = x^5 + x = \frac{p}{q}$$

where a, b, p, q are positive integers. Dividing these gives

$$x^4 + 1 = \frac{m}{n}(x^2 + 1) \text{ where } m = pb, n = qa \text{ are positive integers.}$$

Thus

$$x^4 - \frac{m}{n}x^2 + (1 - \frac{m}{n}) = 0.$$

Then by the quadratic formula we have

$$x^2 = \frac{m \pm \sqrt{m^2 + 4mn - 4n^2}}{2n}. \tag{1}$$

Note that $m^2 + 4mn - 4n^2 \geq 0$ since x (and hence x^2) is assumed to be real. Also $m^2 + 4mn - 4n^2 \neq 0$ because if so then $m = -2n \pm 2n\sqrt{2} = 2n(-1 \pm \sqrt{2})$ and this implies $\frac{m}{n}$ is irrational which is false.

So rewriting (1) gives

$$x^2 = \frac{q \pm w\sqrt{p}}{r} \text{ where } p, q, r, w \text{ are positive integers and } p \text{ is square free.}$$

Thus since we are assuming $x > 0$ we have

$$x = \sqrt{\frac{q \pm w\sqrt{p}}{r}}$$

and by assumption $x^3 + x$ is rational so

$$\sqrt{\frac{q \pm w\sqrt{p}}{r}}\left(1 + \frac{q \pm w\sqrt{p}}{r}\right) = \frac{s}{t}$$

for s, t positive integers. Thus

$$\sqrt{q \pm w\sqrt{p}}\Big((r + q) \pm w\sqrt{p}\Big) = \frac{sr^{\frac{3}{2}}}{t}.$$

Squaring both sides gives

$$(q \pm w\sqrt{p})\left((r+q)^2 + w^2 p \pm 2(r+q)w\sqrt{p}\right) = \frac{s^2 r^3}{t^2}.$$

Thus

$$\left(q\left((r+q)^2 + w^2 p\right) + 2w^2(r+q)p\right) \pm \left(2q(r+q) + (r+q)^2 + w^2 p\right) w\sqrt{p} = \frac{s^2 r^3}{t^2}.$$

However, we now see that the right-hand side is rational but the left-hand side is not because p is square free. Thus it must be that x is rational.