SOLUTION FOR JANUARY 2013

Problem: Let T be a given (non-degenerate) triangle in a plane. Prove there is a constant $c(T)$ with the following property: if a collection of n disks whose areas sum to S entirely contains the sides of T, then $n \geq \frac{c(T)}{S}$.

Solution: The constant $c(T)$ can be chosen to be $\frac{\pi P^2}{4}$ where P is the perimeter of T.

Proof: Let us suppose that the triangle is covered by n disks. Then the sum of the diameters of the n disks must be at least the perimeter, P, of T. Thus $\sum_{i=1}^{n} 2r_i \geq P$ and so

$$\sum_{i=1}^{n} r_i \geq \frac{P}{2}.$$

Now squaring both sides and using the elementary inequality $(\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2$ where the a_i are real we see then that $n \sum_{i=1}^{n} r_i^2 \geq (\sum_{i=1}^{n} r_i)^2 \geq \frac{P^2}{4}$ and thus

$$nS = n \sum_{i=1}^{n} \pi r_i^2 \geq \frac{\pi P^2}{4}$$

where S is the sum of the areas of the disks and therefore

$$n \geq \frac{\pi P^2}{4S} = \frac{c(T)}{S}$$

where $c(T) = \frac{\pi P^2}{4}$.

Further, the constant $c(T) = \frac{\pi P^2}{4}$ cannot be improved because if we take an equilateral triangle ABC of side length a and we take $n = 3$ circles where the first circle has diameter a and has point A and point B as its end points and similarly for the other two circles. Then we see $3a = d_1 + d_2 + d_3 = P$ and so $3(\frac{a}{2}) = \frac{P}{2}$. Squaring and multiplying by π gives $9\pi(\frac{a}{2})^2 = \frac{\pi P^2}{4}$. Now the sum of the areas of the 3 disks is $S = 3\pi(\frac{a}{2})^2$ and so we see $3S = \frac{\pi P^2}{4}$ and thus $n = 3 = \frac{\pi P^2}{4S} = \frac{c(T)}{S}$.