SOLUTION FOR MARCH 2013

Problem

Determine

$$D = \det \begin{pmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{pmatrix}$$

and show it is a product of linear factors.

SOLUTION:

$$D = \det \begin{pmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{pmatrix} = (a-b)^2(a-c)^2(b-c)^2$$

Proof: Recall that the determinant function is linear in each of its columns.

So for example

$$\det \begin{pmatrix} 1+a^2+a^4 & y_1 & z_1 \\ 1+ab+a^2b^2 & y_2 & z_2 \\ 1+ac+a^2c^2 & y_3 & z_3 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{pmatrix} + \det \begin{pmatrix} a^2 & y_1 & z_1 \\ ab & y_2 & z_2 \\ ac & y_3 & z_3 \end{pmatrix} + \det \begin{pmatrix} a^4 & y_1 & z_1 \\ a^2b^2 & y_2 & z_2 \\ a^2c^2 & y_3 & z_3 \end{pmatrix}$$

and similarly for the other columns. Thus the determinant in question will be the sum of 27 determinants!
Denote

$$c_{1,1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};\ c_{1,2} = \begin{pmatrix} a^2 \\ ab \\ ac \end{pmatrix} = a \begin{pmatrix} a \\ b \\ c \end{pmatrix};\ c_{1,3} = \begin{pmatrix} a^4 \\ a^2b^2 \\ a^2c^2 \end{pmatrix} = a^2 \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix};$$

Similarly

$$c_{2,1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};\ c_{2,2} = \begin{pmatrix} ab \\ b^2 \\ bc \end{pmatrix} = b \begin{pmatrix} a \\ b \\ c \end{pmatrix};\ c_{2,3} = \begin{pmatrix} a^2b^2 \\ b^4 \\ b^2c^2 \end{pmatrix} = b^2 \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix};$$

and

$$c_{3,1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};\ c_{3,2} = \begin{pmatrix} ac \\ bc \\ c^2 \end{pmatrix} = c \begin{pmatrix} a \\ b \\ c \end{pmatrix};\ c_{3,3} = \begin{pmatrix} a^2c^2 \\ b^2c^2 \\ c^4 \end{pmatrix} = c^2 \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix};$$

So we see using the multilinearity of the determinant that

$$D = \sum_{i,j,k=1}^{3} \det(c_{1,i}c_{2,j}c_{3,k}).$$

Next note that many of these terms are 0 because they are matrices that have identical columns and we know that the determinant of any matrix with 2 (or more) identical columns is 0. In addition, some of these terms are 0 because one of their columns is a multiple of another column and again the determinant of any such matrix is 0. Therefore the only possibly nonzero determinants are ones where $i, j,$ and k are all different. So it turns out that there are only 6 such matrices which can possibly be nonzero. Thus

$$D = \det(c_{1,1}c_{2,2}c_{3,3}) + \det(c_{1,1}c_{2,3}c_{3,2}) + \det(c_{1,2}c_{2,1}c_{3,3})$$
$$+ \det(c_{1,2}c_{2,3}c_{3,1}) + \det(c_{1,3}c_{2,1}c_{3,2}) + \det(c_{1,3}c_{2,2}c_{3,1}).$$

Let us denote

$$V = \det\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (a-b)(b-c)(c-a). \tag{1}$$

Now the 6 determinats in the formula for D are all multiples of V. Using the fact that you can factor out multiples of a column when calculating determinants we see that for example

$$\det(c_{1,1}c_{2,2}c_{3,3}) = \begin{pmatrix} 1 & ab & a^2c^2 \\ 1 & b^2 & b^2c^2 \\ 1 & bc & c^4 \end{pmatrix} = bc^2 \det\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = bc^2V.$$

In a similar way

$$\det(c_{1,1}c_{2,3}c_{3,2}) = b^2c\begin{pmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{pmatrix} = -cb^2V$$

$$\det(c_{1,2}c_{2,1}c_{3,3}) = -ac^2V$$
$$\det(c_{1,2}c_{2,3}c_{3,1}) = ab^2V$$
$$\det(c_{1,3}c_{2,1}c_{3,2}) = ca^2V$$
$$\det(c_{1,3}c_{2,2}c_{3,1}) = -ba^2V.$$

Adding all of these and using (1) gives

$$(bc^2 - cb^2 - ac^2 + ab^2 + ca^2 - ba^2)V = bc(c-b) + a(b^2-c^2) + a^2(c-b)V$$
$$= (c-b)(bc - a(b+c) + a^2)V = (c-b)(b-a)(c-a)V = (a-b)^2(b-c)^2(a-c)^2.$$

End of proof