SOLUTION FOR APRIL 2013

Determine

$$\int \sqrt{\sqrt{\sqrt{x}+1} - \sqrt{\sqrt{x}-1}}\, dx \text{ for } x > 1. \tag{$*$}$$

SOLUTION:

$$\int \sqrt{\sqrt{\sqrt{x}+1} - \sqrt{\sqrt{x}-1}}\, dx$$

$$= \frac{1}{36}\left(\sqrt{\sqrt{x}+1} - \sqrt{\sqrt{x}-1}\right)^{\frac{9}{2}} + \frac{4}{7}\left(\sqrt{\sqrt{x}+1} - \sqrt{\sqrt{x}-1}\right)^{-\frac{7}{2}} + C.$$

Proof: We first let $u = \sqrt{x}$ and then $2u\, du = dx$. Thus $(*)$ becomes

$$2\int u\sqrt{\sqrt{u+1} - \sqrt{u-1}}\, du. \tag{1}$$

Now let

$$v = \sqrt{u+1} - \sqrt{u-1}. \tag{2}$$

Then

$$dv = -\frac{(\sqrt{u+1} - \sqrt{u-1})}{\sqrt{u^2-1}}\, du$$

and therefore

$$\frac{dv}{v} = \frac{-1}{\sqrt{u^2-1}}\, du. \tag{3}$$

We also observe

$$\frac{v^2}{2} = u - \sqrt{u^2-1} \tag{4}$$

and

$$\frac{2}{v^2} = \frac{1}{u - \sqrt{u^2-1}} = u + \sqrt{u^2-1}. \tag{5}$$

Adding (4)-(5), and dividing by 2 gives

$$u = \frac{1}{v^2} + \frac{v^2}{4} \tag{6}$$

and subtracting (4) from (5), and dividing by 2 gives

$$\sqrt{u^2-1} = \frac{1}{v^2} - \frac{v^2}{4}. \tag{7}$$

Using (7) in (3) gives

$$\left(\frac{v^2}{4} - \frac{1}{v^2}\right)\frac{dv}{v} = du. \tag{8}$$

Now substituting (2), (6) and (8) into (1) gives

$$2\int\left(\frac{v^2}{4}+\frac{1}{v^2}\right)\sqrt{v}\left(\frac{v^2}{4}-\frac{1}{v^2}\right)\frac{dv}{v}=\int\left(\frac{1}{8}v^{\frac{7}{2}}-2v^{-\frac{9}{2}}\right)\,dv$$

$$=\frac{1}{36}v^{\frac{9}{2}}+\frac{4}{7}v^{-\frac{7}{2}}+C=\frac{1}{36}(\sqrt{u+1}-\sqrt{u-1})^{\frac{9}{2}}+\frac{4}{7}(\sqrt{u+1}-\sqrt{u-1})^{-\frac{7}{2}}+C$$

$$=\frac{1}{36}\left(\sqrt{\sqrt{x}+1}-\sqrt{\sqrt{x}-1}\right)^{\frac{9}{2}}+\frac{4}{7}\left(\sqrt{\sqrt{x}+1}-\sqrt{\sqrt{x}-1}\right)^{-\frac{7}{2}}+C$$