PROBLEM OF THE MONTH
OCTOBER 2014 - SOLUTION

Determine:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2} = 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots.$$

Solution:

$$\frac{\ln(2)}{3} + \frac{\pi}{3^{\frac{3}{2}}}.$$

You may recall the formula for a finite geometric series which says for $x \neq 1$:

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}.$$

Replacing x by $-x^3$ gives for $x \neq 1$:

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^{n+1}x^{3n-3} = \frac{1-(-1)^n x^{3n}}{1+x^3}.$$

Integrating on $[0, 1]$ gives:

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{(-1)^{n+1}}{3n-2} = \int_0^1 \frac{1-(-1)^n x^{3n}}{1+x^3}\, dx = \int_0^1 \frac{1}{1+x^3}\, dx + \int_0^1 \frac{(-1)^n x^{3n}}{1+x^3}\, dx. \quad (1)$$

Next we observe since $0 \leq x \leq 1$ that:

$$\left| \int_0^1 \frac{(-1)^n x^{3n}}{1+x^3}\, dx \right| \leq \int_0^1 \frac{x^{3n}}{1+x^3}\, dx \leq \int_0^1 x^{3n}\, dx = \frac{1}{3n+1} \to 0 \text{ as } n \to \infty.$$

Thus we see by taking limits in (1) that:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2} = \int_0^1 \frac{1}{1+x^3}\, dx.$$

Now using partial fractions:

$$\int_0^1 \frac{1}{1+x^3}\, dx = \frac{1}{3} \int_0^1 \left(\frac{1}{1+x} + \frac{-x+2}{x^2-x+1} \right) dx$$

$$= \frac{1}{3} \left(\ln(1+x) - \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)\right) \right) \Big|_0^1$$

$$= \frac{\ln(2)}{3} + \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\ln(2)}{3} + \frac{\pi}{3^{\frac{3}{2}}}.$$