

PROBLEM OF THE MONTH
NOVEMBER 2014 - SOLUTION

Determine:

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x} dx \quad \text{when } a > 0.$$

Solution:

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x} dx = \frac{\pi}{2} \ln(a) \quad \text{when } a > 0.$$

First we rewrite:

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x} dx = \lim_{b \rightarrow \infty} \left(\int_0^b \frac{\tan^{-1}(ax)}{x} dx - \int_0^b \frac{\tan^{-1}(x)}{x} dx \right). \quad (1)$$

Making the change of variables in the first integral:

$$u = ax, du = a dx$$

gives:

$$\int_0^b \frac{\tan^{-1}(ax)}{x} dx = \int_0^{ab} \frac{\tan^{-1}(u)}{u} du.$$

Therefore we can rewrite (1) as:

$$\lim_{b \rightarrow \infty} \left(\int_0^b \frac{\tan^{-1}(ax)}{x} dx - \int_0^b \frac{\tan^{-1}(x)}{x} dx \right) = \lim_{b \rightarrow \infty} \int_b^{ab} \frac{\tan^{-1}(x)}{x} dx.$$

Now we make another change of variables:

$$x = by, dx = b dy$$

which gives:

$$\lim_{b \rightarrow \infty} \int_b^{ab} \frac{\tan^{-1}(x)}{x} dx = \lim_{b \rightarrow \infty} \int_1^a \frac{\tan^{-1}(by)}{y} dy. \quad (2)$$

Finally, since $\tan^{-1}(by) \rightarrow \frac{\pi}{2}$ uniformly on $[1, \infty) \cup [a, \infty)$ then from (1)-(2):

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x} dx = \lim_{b \rightarrow \infty} \int_1^a \frac{\tan^{-1}(by)}{y} dy = \frac{\pi}{2} \int_1^a \frac{1}{y} dy = \frac{\pi}{2} \ln(a).$$