SOLUTION FOR DECEMBER 2014

Winner - Mark Fincher

PROBLEM from December 2014

Determine:

$$\sum_{k=0}^{n} \binom{n}{k} k^3$$

SOLUTION:

$$\sum_{k=0}^{n} \binom{n}{k} k^3 = n^2(n+3)2^{n-3}$$

Proof: From the binomial theorem we know that

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Differentiating gives

$$n(x+1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}.$$

Now multiply by x and differentiate again to obtain

$$n(x+1)^{n-2}(nx+1) = \sum_{k=0}^{n} \binom{n}{k} k^2 x^{k-1}.$$

Again multiply by x and differentiate to obtain

$$n2^{n-3}(n^2x^2 + (3n-1)x + 1) = \sum_{k=0}^{n} \binom{n}{k} k^3 x^{k-1}.$$

Finally let $x = 1$ to obtain

$$n^2(n+3)2^{n-3} = \sum_{k=0}^{n} \binom{n}{k} k^3.$$