JANUARY 2014 SOLUTION

Problem: Determine the triangle of largest area that can be inscribed in the ellipse

$$x^2 + xy + y^2 = 1.$$

Solution: The triangle of largest area is of area $\frac{3}{4}$.

Proof First we observe the well-known fact that any ellipse can be placed in standard form by a rotation of coordinates so that in the new coordinate system the equation for the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In particular, if we rotate by 45^0 and let

$$x' = \frac{x+y}{\sqrt{2}}$$

$$y' = \frac{x-y}{\sqrt{2}}$$

then we see

$$1 = x^2 + xy + y^2 = \frac{3}{2}x'^2 + \frac{1}{2}y'^2 = \frac{x'^2}{\left(\sqrt{\frac{2}{3}}\right)^2} + \frac{y'^2}{\left(\sqrt{\frac{1}{2}}\right)^2}$$

Next we show that the largest triangle incribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has area

$$\frac{3\sqrt{3}}{4}ab.$$

Then using $a = \sqrt{\frac{2}{3}}$ and $b = \sqrt{\frac{1}{2}}$ we see that the triangle of largest area that we are looking for in this problem has area $\frac{3\sqrt{3}}{4}\sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}} = \frac{3}{4}$.

To show the area of the largest triangle in the ellipse is $\frac{3\sqrt{3}}{4}ab$, we first parametrize the ellipse by $(a\cos(\theta), b\sin(\theta))$ and then we choose 3 points on the ellipse $(a\cos(\theta_1), b\sin(\theta_1))$, $(a\cos(x), b\sin(x))$ and $(a\cos(y), b\sin(y))$ with $\theta_1 < x < y < \theta_1 + 2\pi$. Let

$$v_1 = (a(\cos(x) - \cos(\theta_1)), b(\sin(x) - \sin(\theta_1)))$$

and

$$v_2 = (a(\cos(y) - \cos(\theta_1)), b(\sin(y) - \sin(\theta_1)))\,.$$

Then we know that the area of the parallelogram determined by vectors v_1 and v_2 is $|\det(v_1\, v_2)|$ and therefore the area of the triangle determined by v_1 and v_2 is

$$A = \frac{1}{2}|\det(v_1\, v_2)| = \frac{1}{2}ab|\Big((\cos(x){-}\cos(\theta_1))(\sin(y){-}\sin(\theta_1)){-}(\sin(x){-}\sin(\theta_1))(\cos(y){-}\cos(\theta_1))\Big)|$$

and after multiplying this out and using the subtraction formula for the sine function yields

$$A = \frac{1}{2}ab\Big(|\sin(y-x) + \sin(x-\theta_1) - \sin(y-\theta_1)|\Big).$$

We next show that the expression under the absolute value sign in the above equation is greater than or equal to 0 and therefore the absolute value signs in the above equation are not necessary. Thus

$$A = \frac{1}{2}ab\Big(\sin(y-x) + \sin(x-\theta_1) - \sin(y-\theta_1)\Big). \tag{1}$$

To see this, using the sum-to-product formulas it follows that

$$A = 2ab\sin\left(\frac{y-x}{2}\right)\sin\left(\frac{y-\theta_1}{2}\right)\sin\left(\frac{x-\theta_1}{2}\right). \tag{1A}$$

Since $\theta \le x \le y \le \theta_1 + 2\pi$ it follows that $0 \le \frac{y-x}{2} \le \pi$, $0 \le \frac{y-\theta_1}{2} \le \pi$, and $0 \le \frac{x-\theta_1}{2} \le \pi$, and so it follows that the three terms in (1A) are all nonnegative.

Next we want to find the maximum of $A(x,y)$ on the triangular region $[\theta_1, \theta_1+2\pi] \times [x, \theta_1+2\pi]$ where $\theta_1 \le x \le \theta_1 + 2\pi$ and $x \le y \le \theta_1 + 2\pi$. We observe that $A(x,y)$ is 0 on the boundary of this region and nonnegative in the interior and so the maximum occurs at an interior critical point. We will see there is only one critical point and then that the critical point is a maximum.

We next see

$$A_x = \frac{1}{2}ab\Big(-\cos(y-x) + \cos(x-\theta_1)\Big)$$

and

$$A_y = \frac{1}{2}ab\Big(\cos(y-x) - \cos(y-\theta_1)\Big).$$

Thus setting $A_x = 0$ and $A_y = 0$ yields

$$\cos(y-x) = \cos(x-\theta_1) \tag{2}$$

and

$$\cos(y-x) = \cos(y-\theta_1). \tag{3}$$

Equation (2) yields $y-x = x-\theta_1$ or $y-x = 2\pi-(x-\theta_1)$. The second equation gives $y = 2\pi+\theta_1$ and this is not in the domain so we see that (2) implies

$$y = 2x - \theta_1. \tag{4}$$

Equation (3) yields $y - x = y - \theta_1$ or $y - x = 2\pi - (y - \theta_1)$. The first of these is not possible so we must have $y - x = 2\pi - (y - \theta_1)$ and thus

$$2y - x = 2\pi + \theta_1. \tag{5}$$

Substituting (4) into (5) gives $4x - \theta_1 - x = 2\pi + \theta_1$ and thus

$$x = \theta_1 + \frac{2\pi}{3}.$$

Then returning to (4) we obtain

$$y = \theta_1 + \frac{4\pi}{3}.$$

Thus we see θ_1, x, and y are all separated by an angle of $\frac{2\pi}{3}$. Substituting this into (1) then gives

$$\begin{aligned} A &= \frac{1}{2}ab\Big(\sin(\frac{2\pi}{3}) + \sin(\frac{2\pi}{3}) - \sin(\frac{4\pi}{3})\Big) \\ &= \frac{1}{2}ab\Big(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - (-\frac{\sqrt{3}}{2})\Big) = \frac{3\sqrt{3}}{4}ab. \end{aligned}$$