MARCH 2014 SOLUTION

Consider triangle BAC with $\angle BAC = 40^0$. Let $AB = 5$ and $AC = 3$. Let E be on AC and D on AB. Determine the minimum of $BE + DE + CD$.

Solution The minimum is

$$\sqrt{34 + 30\cos(40^0)(3 - 4\cos^3(40^0))}.$$

Proof Let $d = 40^0$. Let $AD = x \geq 0$, $AE = y \geq 0$. Denote $AB = P > 0$ and $AC = Q > 0$. Then by the law of cosines we have

$$DE = \sqrt{x^2 + y^2 - 2xy\cos d},$$

$$BE = \sqrt{y^2 + P^2 - 2yP\cos d},$$

$$CD = \sqrt{x^2 + Q^2 - 2xQ\cos d}.$$

Later we will specialize to the case $AB = P = 5$ and $AC = Q = 3$.

Let

$$f(x, y) = \sqrt{x^2 + y^2 - 2xy\cos d} + \sqrt{y^2 + P^2 - 2yP\cos d} + \sqrt{x^2 + Q^2 - 2xQ\cos d}. \quad (1)$$

We want to find the minimum of $f(x, y)$ over the rectangle $[0, P] \times [0, Q]$.

We first look for any interior critical points, i.e. points where $f_x = f_y = 0$. Then

$$f_x = \frac{x - y\cos d}{\sqrt{x^2 + y^2 - 2xy\cos d}} + \frac{x - Q\cos d}{\sqrt{x^2 + Q^2 - 2xQ\cos d}} = 0.$$

Bringing one of these equations to the other side and squaring gives

$$\frac{(x - y\cos d)^2}{(x - y\cos d)^2 + y^2\sin^2 d} = \frac{(x - Q\cos d)^2}{(x - Q\cos d)^2 + Q^2\sin^2 d}.$$

Inverting and simplifying gives

$$\frac{y}{x - y\cos d} = \frac{-Q}{x - Q\cos d}. \quad (2)$$

This then implies

$$x = \frac{2Qy\cos d}{Q + d}. \quad (3)$$

Similarly setting $f_y = 0$ we obtain

$$\frac{x}{y - x\cos d} = \frac{-P}{y - P\cos d} \quad (4)$$

and

$$y = \frac{2Px\cos d}{P + x}. \tag{5}$$

Solving (3) and (5) we see that

$$x = \frac{PQ(4\cos^2 d - 1)}{Q + 2P\cos d}, \quad y = \frac{PQ(4\cos^2 d - 1)}{P + 2Q\cos d}. \tag{6}$$

Now substituting (4) into the second term on the right-hand side in (1) gives

$$y^2+P^2-2yP\cos d = (y-P\cos d)^2+P^2\sin^2 d = \frac{P^2}{x^2}\left((y - x\cos d)^2 + x^2\sin^2 d\right) = \frac{P^2}{x^2}(x^2+y^2-2xy\cos d). \tag{7}$$

Similarly substituting (2) into the third term on the right-hand side in (1) gives

$$x^2+Q^2-2xQ\cos d = (x-Q\cos d)^2+Q^2\sin^2 d = \frac{Q^2}{y^2}\left((x - y\cos d)^2 + y^2\sin^2 d\right) = \frac{Q^2}{y^2}(x^2+y^2-2xy\cos d). \tag{8}$$

Therefore substituting the square root of equation (7) and (8) into (1) we see that at the critical point in (6) we have

$$f(x, y) = \left(1 + \frac{P}{x} + \frac{Q}{y}\right)\sqrt{x^2 + y^2 - 2xy\cos d}. \tag{9}$$

Next using (6) we obtain

$$\frac{P}{x} + \frac{Q}{y} = \frac{2PQ + 2(P^2 + Q^2)\cos d}{PQ(4\cos^2 d - 1)}$$

and thus we see

$$1+\frac{P}{x}+\frac{Q}{y} = \frac{2PQ + 2(P^2 + Q^2)\cos d + PQ(4\cos^2 d - 1)}{PQ(4\cos^2 d - 1)} = \frac{PQ + 2(P^2 + Q^2)\cos d + 4PQ\cos^2 d}{PQ(4\cos^2 d - 1)}$$

$$= \frac{(P + 2Q\cos d)(Q + 2P\cos d)}{PQ(4\cos^2 d - 1)}. \tag{10}$$

Also using (6) we see

$$x^2+y^2-2xy\cos d = P^2Q^2(4\cos^2 d-1)^2\left(\frac{1}{(P + 2Q\cos d)^2} + \frac{1}{(Q + 2P\cos d)^2} - \frac{2\cos d}{(P + 2Q\cos d)(Q + 2P\cos d)}\right).$$

Thus

$$\sqrt{x^2 + y^2 - 2xy\cos d} = \tag{11}$$

$$= PQ(4\cos^2 d - 1)\frac{\sqrt{(Q + 2P\cos d)^2 + (P + 2Q\cos d)^2 - 2\cos d(P + 2Q\cos d)(Q + 2P\cos d)}}{(P + 2Q\cos d)(Q + 2P\cos d)}.$$

Thus at (6) we see that

$$f(x, y) = \sqrt{(Q + 2P\cos d)^2 + (P + 2Q\cos d)^2 - 2\cos d(P + 2Q\cos d)(Q + 2P\cos d)}$$

$$= \sqrt{(P^2 + Q^2) + (2PQ\cos d)(3 - 4\cos^2 d)}. \qquad (12)$$

Next we investigate f on the boundary of the region $[0, P] \times [0, Q]$.

Note that $f(0, 0) = P + Q$. Also note that

$$f(x, 0) = x + P + \sqrt{x^2 + Q^2 - 2xQ\cos d} \text{ and } f(0, y) = y + Q + \sqrt{y^2 + P^2 - 2yP\cos d}$$

and that

$$f_x(x, 0) = 1 + \frac{(x - Q\cos d)}{\sqrt{(x - Q\cos d)^2 + Q^2\sin^2 d}} > 0 \text{ on } [0, P].$$

Similarly

$$f_y(0, y) = 1 + \frac{(y - P\cos d)}{\sqrt{(y - P\cos d)^2 + P^2\sin^2 d}} > 0 \text{ on } [0, Q].$$

Since f is continuous we see that $f(x, y)$ has a possible minimum at $(0, 0)$ but not at any point on $[x, 0]$ with $0 < x \leq P$ nor at any point on $[0, y]$ with $0 < y \leq Q$.

Next we see

$$f(P, y) = 2\sqrt{y^2 + P^2 - 2yP\cos d} + \sqrt{P^2 + Q^2 - 2PQ\cos d}$$

and

$$f_y(P, y) = \frac{2(y - P\cos d)}{\sqrt{y^2 + P^2 - 2yP\cos d}}$$

and also note that $f_y(P, y) < 0$ on $[0, Q]$ since by assumption $y - P\cos d \leq Q - P\cos d < 0$ i.e. $3 - 5\cos(40^0) < 0$.

Also note that $f(P, Q) = 3\sqrt{P^2 + Q^2 - 2PQ\cos d} > P + Q = f(0, 0)$ (true since $\cos(40^0) < 7/9$) so the min is not at (P, Q).

Next we see

$$f(x, Q) = 2\sqrt{x^2 + Q^2 - 2xQ\cos d} + \sqrt{P^2 + Q^2 - 2PQ\cos d}$$

and

$$f_x(x, Q) = \frac{2(x - Q\cos d)}{\sqrt{x^2 + Q^2 - 2xQ\cos d}}.$$

Note $f_x(x, Q) = 0$ when $x = Q\cos d$ and by assumption $Q\cos d < P$, i.e. $3\cos d < 5$.
Thus we see that the minimum of f on $[0, P] \times [0, Q]$ occurs either at

$$(0, 0), \left(\frac{PQ(4\cos^2 d - 1)}{Q + 2P\cos d}, \frac{PQ(4\cos^2 d - 1)}{P + 2Q\cos d}\right), \text{ or at } (Q\cos d, Q).$$

Therefore using (12) we see the minimum is the smallest of the three numbers

$$f(0, 0),\ f\left(\frac{PQ(4\cos^2 d - 1)}{Q + 2P\cos d}, \frac{PQ(4\cos^2 d - 1)}{P + 2Q\cos d}\right), \text{ and } f(Q\sin d, Q)$$

i.e.

$$P+Q, \sqrt{(P^2+Q^2)+(2PQ\cos d)(3-4\cos^2 d)}, \text{ and } 2Q\sin d + \sqrt{P^2+Q^2-2PQ\cos d}.$$

Next note that $\cos d(3-4\cos^2 d) < 1$ because this is equivalent to $(\cos d+1)(2\cos^2 d-1)^2 = 4\cos^3 d - 3\cos d + 1 > 0$ and this is true so

$$\sqrt{(P^2+Q^2)+(2PQ\cos d)(3-4\cos^2 d)} < \sqrt{P^2+Q^2+2PQ} = P+Q.$$

Thus the minimum does not occur at $(0,0)$.

So finally we need to determine the smaller of the two numbers

$$\sqrt{(P^2+Q^2)+(2PQ\cos d)(3-4\cos^2 d)}, \text{ and } 2Q\sin d + \sqrt{P^2+Q^2-2PQ\cos d}. \tag{13}$$

Squaring both of these then it follows we need to determine the smaller of

$$2P\cos d\sin d \quad \text{and} \quad Q\sin d + \sqrt{P^2+Q^2-2PQ\cos d}.$$

Substituting $P=5$ and $Q=3$ reduces the problem to the determining the smaller of

$$\sin d(10\cos d - 3) \quad \text{and} \quad \sqrt{34-30\cos d}.$$

Squaring again this reduces to determining the smaller of

$$91\cos^2 d + 6\cos^3 d \text{ and } 25 + 30\cos d + 100\cos^4 d$$

Finally using that $\cos d = \cos(40^0) \approx .75$ yields the smaller of the two numbers is the one on the left and going back to (13) we see the smaller of the two numbers is on the left hand side of (13). Thus the minimum is

$$\sqrt{34+30\cos(40^0)(3-4\cos^3(40^0))}.$$