APRIL 2014 SOLUTION

Let A be a real 4×4 matrix with characteristic polynomial $p(\lambda) = \det(\lambda I - A) = \lambda^4 - S_1\lambda^3 + S_2\lambda^2 - S_3\lambda + S_4$. Suppose there is a real matrix S such that $S^2 = A$. Prove that

$$1 + S_1 + S_2 + S_3 + S_4 \geq 0.$$

Proof Since $S^2 = A$ we see that $\det A = \det(S^2) = (\det S)^2 \geq 0$ (since S is a real matrix).

Case 1 $\det A > 0$.

We notice that the statement we are trying to prove is that $p(-1) \geq 0$. We suppose that the result is false. That is, we assume

$$p(-1) = 1 + S_1 + S_2 + S_3 + S_4 < 0.$$

Also $p(0) = S_4 = \det A > 0$. Now since $p(\lambda) \to \infty$ as $\lambda \to \pm\infty$ then we see that $p(\lambda)$ has at least two negative zeros, say $\lambda_2 < \lambda_1 < 0$. Now since $\lambda I - A = (\sqrt{\lambda} I - S)(\sqrt{\lambda} I + S)$ we see that either $i\sqrt{-\lambda_1}$ is an eigenvalue for S or $-i\sqrt{-\lambda_1}$ is an eigenvalue for S. In fact, they are both eigenvalues for S because suppose $Sw = i\sqrt{-\lambda_1}w$ with $w \neq 0$. Then w is not a real vector because if so then the left-hand side of the previous equation is real and the right-hand side is not. Then since S is real

$$S\overline{w} = \overline{Sw} = \overline{i\sqrt{-\lambda_1}w} = -i\sqrt{-\lambda_1}\overline{w}.$$

Similarly if $-i\sqrt{-\lambda_1}$ is an eigenvalue then so is $i\sqrt{-\lambda_1}$.
In a similar way $i\sqrt{-\lambda_2}$ and $-i\sqrt{-\lambda_2}$ are both eigenvalues for S. Thus we see that S has 4 eigenvalues and thus

$$\det(\mu I - S) = (\mu - i\sqrt{-\lambda_1})(\mu + i\sqrt{-\lambda_1})(\mu - i\sqrt{-\lambda_2})(\mu + i\sqrt{-\lambda_2}) = (\mu^2 + (-\lambda_1))(\mu^2 + (-\lambda_2)).$$

Similarly $-S$ has 4 eigenvalues and

$$\det(\mu I + S) = (\mu^2 + (-\lambda_1))(\mu^2 + (-\lambda_2)).$$

Finally,

$$\det(\mu^2 I - A) = \det(\mu I - S)\det(\mu I + S) = (\mu^2 + (-\lambda_1))^2(\mu^2 + (-\lambda_2))^2$$

And thus

$$p(\lambda) = \det(\lambda I - A) = (\lambda + (-\lambda_1))^2(\lambda + (-\lambda_2))^2 \geq 0$$

but this contradicts that $p(-1) < 0$. Thus it must be that $p(-1) \geq 0$.
This completes the proof in the case $\det A > 0$.

Case 2 $\det A = 0$.

Since $S^2 = A$ it follows that $(\det S)^2 = \det S^2 = \det A = 0$ and so $\det S = 0$. Now again let us suppose $p(-1) < 0$. Then we know that $p(\lambda)$ has a negative eigenvalue $\lambda_1 < 0$ and a 0 eigenvalue. As earlier we see that S has eigenvalues $\pm i\sqrt{-\lambda_1}$. And also 0 is an eigenvalue so

S has at least three eigenvalues. Since S is a 4 by 4 matrix then we know that $\det(\mu I - S)$ is of degree 4. Also since S is real and has two complex roots and one real root then it must be that S has another real eigenvalue, say b, with b possibly equal to 0. Thus,

$$\det(\mu I - S) = \mu(\mu^2 + (-\lambda_1))(\mu - b).$$

Notice that the 4 eigenavlues of $-S$ are $i\sqrt{-\lambda_1}$, $-i\sqrt{-\lambda_1}$, 0, and $-b$ so that $\det(\mu I - S) = \mu(\mu^2 + (-\lambda_1))(\mu + b)$ and therefore

$$\det(\mu^2 I - A) = \det(\mu^2 I - S^2) = \det(\mu^2 I - A) = \det(\mu I - S)\det(\mu I + S) = \mu^2(\mu^2 + (-\lambda_1))^2(\mu^2 - b^2).$$

Thus

$$p(\lambda) = \det(\lambda I - A) = \lambda(\lambda + (-\lambda_1))^2(\lambda - b^2).$$

Therefore

$$0 > p(-1) = (-1)(-1 + (-\lambda_1))^2(-1 - b^2) = (-1 + (-\lambda_1))^2(1 + b^2) \geq 0$$

yielding a contradiction. Thus in this situation as well we see that $p(-1) < 0$ leads to a contradiction and thus $p(-1) \geq 0$.