SOLUTION FOR NOVEMBER 2015 PROBLEM

Find all real solutions of:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n}}{(2n)!} = 0.$$

Solution:
There are no real solutions!

(One might suspect this since $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n}}{(2n)!}$ is a partial sum for e^x and $e^x > 0$).

If we denote:

$$s_n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

then it follows that

$$s_n' = s_{n-1}.$$

We now show that $s_{2n} = 0$ has no real solutions by induction. First, $s_0 \equiv 1 > 0$ and so $s_0 = 0$ has no real solutions. Now we assume that $s_{2n} \neq 0$ and we try to show $s_{2n+2} \neq 0$. Since $s_{2n}(0) = 1 > 0$ it follows that $s_{2n} > 0$ for all x. Notice now that $s'_{2n+1} = s_{2n} > 0$ and so $s'_{2n+1} > 0$ for all x - i.e. s_{2n+1} is strictly increasing for all x. Since s_{2n+1} is a polynomial of odd degree we know that it must have at least one real zero and since we now know that s_{2n+1} is strictly increasing we see that s_{2n+1} has exactly one zero, p. Further, notice that $s_{2n+1}(x) > 0$ when $x \geq 0$ and so it must be that $p < 0$. We also have that $s'_{2n+2} = s_{2n+1}$ and since s_{2n+1} has exactly one zero we see that s_{2n+2} has p as a critical point and this must be a minimum. Also note that $s_{2n+2} = s'_{2n+2} + \frac{x^{2n+2}}{(2n+2)!}$ so that finally we see that $s_{2n+2}(p) = s'_{2n+2}(p) + \frac{p^{2n+2}}{(2n+2)!} = 0 + \frac{p^{2n+2}}{(2n+2)!} > 0$ since $p < 0$. Thus s_{2n+2} has exactly one critical point, p, which is a minimum and at this point $s_{2n+2}(p) > 0$ and so $s_{2n+2} > 0$ for all x. So we see by induction that $s_{2n+2}(x) > 0$ for all x and so it follows that $s_{2n} > 0$ for all x and for all positive integers n.