

### SOLUTION FOR DECEMBER 2015

Determine if the following series converges:

$$1 + \frac{1}{2} \left( \frac{19}{7} \right) + \frac{2!}{3^2} \left( \frac{19}{7} \right)^2 + \frac{3!}{4^3} \left( \frac{19}{7} \right)^3 + \cdots$$

**SOLUTION:** The series does indeed converge.

Notice that if we denote  $x = \frac{19}{7}$  then we can rewrite the series as:

$$1 + \frac{1}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^n} x^n.$$

Applying the ratio test we see that:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{\frac{(n+1)!|x|^{n+1}}{(n+2)^{n+1}}}{\frac{n!|x|^n}{(n+1)^n}} = \frac{(n+1)(n+1)^n|x|}{(n+2)^{n+1}} \\ &= \frac{(n+1)^{n+1}|x|}{(n+2)^{n+1}} = \left( 1 - \frac{1}{n+2} \right)^{n+1} |x| = \frac{\left( 1 - \frac{1}{n+2} \right)^{n+2}}{\left( 1 - \frac{1}{n+2} \right)} |x|. \end{aligned}$$

You might recall from calculus that:

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n = e^x$$

and in particular that:

$$\lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n} \right)^n = e^{-1}.$$

And so it follows that:

$$\lim_{n \rightarrow \infty} \frac{\left( 1 - \frac{1}{n+2} \right)^{n+2}}{\left( 1 - \frac{1}{n+2} \right)} |x| = e^{-1} |x|.$$

Thus the above series converges when:

$$e^{-1}|x| < 1, \text{ i.e. when } |x| < e$$

and since  $x = \frac{19}{7} < e$  then we see that the given series converges by the ratio test. (Note that  $\frac{19}{7} = 2.714\dots$  and  $e = 2.718\dots$ ).