SOLUTION FOR JANUARY 2015

Find all continuous nonnegative functions which satisfy:

$$f(x+t) = f(x) + f(t) + 2\sqrt{f(x)}\sqrt{f(t)} \text{ for } x \geq 0, t \geq 0. \tag{1}$$

SOLUTION:

$$f(x) = cx^2 \text{ where } c \geq 0.$$

Proof: First, notice that we may rewrite (1) as:

$$f(x+t) = \left(\sqrt{f(x)} + \sqrt{f(t)}\right)^2$$

and so since $f(x)$ is nonnegative:

$$\sqrt{f(x+t)} = \sqrt{f(x)} + \sqrt{f(t)}.$$

Denoting $g(x) = \sqrt{f(x)}$ we see then that:

$$g(x+t) = g(x) + g(t) \text{ for } x \geq 0, t \geq 0. \tag{2}$$

Substituting $x = t = 0$ into (2) we see that:

$$g(0) = 2g(0)$$

and so $g(0) = 0$. Substituting $x = t$ into (2) gives:

$$g(2t) = 2g(t).$$

It follows by induction that:

$$g(mt) = mg(t) \tag{3}$$

for every positive integer m. Letting $t = \frac{1}{m}$ in (3) gives:

$$g\left(\frac{1}{m}\right) = \frac{1}{m}g(1). \tag{4}$$

Letting $t = \frac{1}{n}$ in (3) and (4) gives:

$$g\left(\frac{m}{n}\right) = mg\left(\frac{1}{n}\right) = \frac{m}{n}g(1).$$

Thus

$$g(r) = g(1)r$$

for every nonnegative rational number r. Finally since the rational numbers are dense in the set of real numbers and since we assumed f is continuous, it follows that g is continuous and therefore it must be that:

$$g(x) = g(1)x$$

for every nonnegative real number x. Denoting $d = g(1)$ we obtain:

$$\sqrt{f(x)} = g(x) = dx$$

for every nonnegative real number x. Hence:

$$f(x) = cx^2$$

for every nonnegative real number x where $c = d^2 \geq 0$.