SOLUTION FOR APRIL 2015

Let $x \geq 0, y \geq 0$, and $z \geq 0$. Find all solutions of:

$$x^{1/3} - y^{1/3} - z^{1/3} = 16$$
$$x^{1/4} - y^{1/4} - z^{1/4} = 8$$
$$x^{1/6} - y^{1/6} - z^{1/6} = 4.$$

SOLUTION: The only solution is:

$$x = 4096, y = 0, z = 0.$$

We first introduce new variables: $a = x^{1/12}, b = y^{1/12}$, and $c = z^{1/12}$ so that the above equations become:

$$a^4 - b^4 - c^4 = 16$$
$$a^3 - b^3 - c^3 = 8$$
$$a^2 - b^2 - c^2 = 4.$$

Rewriting the first and third equations we see that:

$$a^4 = 16 + b^4 + c^4$$
$$a^2 = 4 + b^2 + c^2.$$

Squaring this second equation and equating it to the first equation gives:

$$16 + b^4 + c^4 = a^4 = 16 + b^4 + c^4 + 8b^2 + 8c^2 + 2b^2c^2.$$

Thus we see that:

$$4b^2 + b^2c^2 + 4c^2 = 0.$$

Since each of these terms is nonnegative we see that the only solution of this equation is $b = c = 0$. Substituting this into the original equations gives:

$$a^4 = 16$$
$$a^3 = 8$$
$$a^2 = 4$$

and the only nonnegative solution of this is $a = 2$. Returning to the original variables x, y, and z we see that the only solution of the original system of equations is:

$$x = 2^{12} = 4096, y = z = 0.$$